CONCENTRATION AT SUB-MANIFOLDS FOR AN ELLIPTIC DIRICHLET PROBLEM NEAR HIGH CRITICAL EXPONENTS

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Abstract. Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the equation $\Delta u + u^{\frac{n-k+2}{n-k-2}-\varepsilon} = 0$ in Ω, under zero Dirichlet boundary condition, where ε is a small positive parameter. We assume that there is a k-dimensional closed, embedded minimal sub-manifold K of $\partial\Omega$, which is non-degenerate, and along which a certain weighted average of sectional curvatures of $\partial\Omega$ is negative. Under these assumptions, we prove existence of a sequence $\varepsilon = \varepsilon_j$ and a solution u_ε which concentrate along K, as $\varepsilon \to 0^+$, in the sense that

$$|\nabla u_\varepsilon|^2 \rightharpoonup S_{n-k}^{\frac{n-k}{2}} \delta_K \quad \text{as } \varepsilon \to 0$$

where δ_K stands for the Dirac measure supported on K and S_{n-k} is an explicit positive constant. This result generalizes the one obtained in [17], where the case $k = 1$ is considered.

1. Introduction and statement of main results

Consider the following nonlinear problem known as the Lane-Emden-Fowler problem ([21])

$$\begin{cases} \Delta u + u^p = 0, & u > 0 \quad \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$ and $p > 1$. When the exponent p is subcritical ($1 < p < \frac{n+2}{n-2}$), compactness of Sobolev's embedding yields a solution as a minimizer of the variational problem

$$S(p) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}. \tag{1.2}$$

For the case $p \geq \frac{n+2}{n-2}$, this approach fails and essential obstructions to existence arise: Pohozaev [32] found that no solution to (1.1) exists if the domain is star-shaped. In contrast, Kazdan and Warner [23] observed that if Ω is a symmetric annulus then compactness holds for any $p > 1$ within the class of radial functions, and a solution can again always be found by the above minimizing procedure. Compactness in the minimization is also restored, without symmetries, by the addition of suitable linear perturbations exactly at the critical exponent $p = \frac{n+2}{n-2}$, as established by Brezis and Nirenberg [8].

If $p \geq \frac{n+2}{n-2}$, the topology and geometry of the domain play a crucial role for the solvability of the above problem; indeed, for $p = \frac{n+2}{n-2}$, Bahri and Coron [3] proved the existence of solution to (1.1) when the topology of Ω is non-trivial in suitable sense. For powers larger than critical direct use of variational arguments seems hopeless, and one needs more general arguments to get solvability. The presence of nontrivial topology turns out to be not sufficient to get solvability in the supercritical situation $p > \frac{n+2}{n-2}$. In fact, for $n \geq 4$ Passaseo [31] exhibited a domain constituted by a thin tubular neighborhood of a copy of the sphere $\mathbb{S}^{n-2}$ embedded in $\mathbb{R}^n$ for which a Pohozaev-type identity yields that no solution exists if $p \geq \frac{n+1}{n-3}$ (*the so-called second critical exponent*).

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In this paper we consider the case when p is below but sufficiently close to the k-th critical exponent (the Sobolev critical exponent in dimension $n-k$) defined as $\frac{n-k+2}{n-k-2}$, with $0 \le k \le n-1$. Namely we consider the following problem

$$\begin{cases} \Delta u + u^{\frac{n-k+2}{n-k-2}-\varepsilon} = 0, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ is a small parameter. Assuming that $\partial\Omega$ contains a closed minimal non-degenerate sub-manifold K of dimension k along which a certain weighted average of sectional curvatures of $\partial\Omega$ is negative, we find a solution to (1.1) which concentrates as p approaches $\frac{n+2-k}{n-2-k}$ (as ε tends to 0^+) in a sense to be determined later. Before we state our main result, let us recall some previous works in the cases $k = 0$ (point bubbling) and $k = 1$ (line bubbling).

The case $k = 0$ has been extensively considered in the literature, see for instance [7, 22, 33, 20] and some references therein. It has been proven the existence of *bubbling solutions* around special points of the domain, which resemble a sharp extremal of the best Sobolev constant in $\mathbb{R}^n$

$$S_n := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2}{\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}.$$

The behavior of a solution u_ε which minimizes $S(p)$ in (1.2) for $p = p_\varepsilon = \frac{n+2}{n-2} - \varepsilon$, is given by

$$u_\varepsilon(x) = \mu_\varepsilon^{-\frac{n-2}{2}} w_n(\mu_\varepsilon^{-1}(x - x_\varepsilon)) + o(1), \qquad \mu_\varepsilon \sim \varepsilon^{\frac{1}{n-2}},$$

as $\varepsilon \to 0^+$, where w_n is the *standard bubble*,

$$w_n(x) = \left(\frac{c_n}{1+|x|^2}\right)^{\frac{n-2}{2}}, \qquad c_n = (n(n-2))^{\frac{1}{n-2}}, \tag{1.4}$$

a radial solution of

$$\Delta w + w^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n \tag{1.5}$$

corresponding to an extremal for S_n, [2, 34]. The blow-up point x_ε approaches (up to a subsequence) a harmonic center x_0 of Ω, namely a minimizer for Robin's function of the domain, the diagonal of the regular part of Green's function. The solution concentrates as a Dirac mass at x_0, namely

$$|\nabla u_\varepsilon|^2 \rightharpoonup S_n^{\frac{n}{2}} \delta_{x_0} \quad \text{as } \varepsilon \to 0 \tag{1.6}$$

in the sense of measures. We also refer to [4, 12] and to the survey [16] for related results on construction of point-bubbling solutions for problems near the critical exponent.

The case $k = 1$ has been studied by del Pino-Musso-Pacard [17]. They proved that given a closed non-degenerate geodesic Γ on $\partial\Omega$, which has globally *negative curvature* and assuming that a non-resonance condition holds, then for $n \ge 8$, problem (1.3) with $k = 1$ has a solution u_ε that satisfies

$$|\nabla u_\varepsilon|^2 \rightharpoonup S_{n-1}^{\frac{n-1}{2}} \delta_\Gamma$$

as $\varepsilon \to 0$ in the sense of measures, where δ_Γ is the Dirac measure supported on the curve Γ.

This result shows that line-bubbling phenomenon is conceptually quite different to point bubbling. In fact, point concentration is determined by global information on the domain encoded in Green's function, while only local structure of the domain near the curve Γ is relevant to the line-bubbling. This is a typical phenomenon for concentration on positive dimensional sets. We point out that the case $k \ge 2$ under some symmetric assumptions on the domain was studied by Ackermann-Clapp-Pistoia in [1], see also [10] for some related issues.

The purpose of this paper is to study existence of positive solutions to Problem (1.3) when Ω is a non symmetric domain in the general case $1 \le k \le n-1$. Before we state our result we need to introduce the following notations:

Let $q \in K$. We denote by $T_q\partial\Omega$ the tangent space to $\partial\Omega$ at the point q. We consider the *shape operator* $\mathtt{L} : T_q\partial\Omega \to T_q\partial\Omega$ defined as

$$\mathtt{L}[e] := -\nabla_e \nu(q)$$

where $\nabla_e \nu(q)$ is the directional derivative of the vector field ν in the direction e. Let us consider the orthogonal decomposition

$$T_q\partial\Omega = T_qK \oplus N_qK$$

where N_qK stands for the normal bundle of K. We choose orthonormal bases $(e_a)_{a=1,\dots,k}$ of T_qK and $(e_i)_{i=k+1,\dots,n-1}$ of N_qK.

Let us consider the $(n-1)\times(n-1)$ matrix $H(q)$ representative of $\mathtt{L}$ in these bases, namely

$$H_{\alpha\beta}(q) = e_\alpha \cdot \mathtt{L}[e_\beta].$$

This matrix also represents the second fundamental form of $\partial\Omega$ at q in this basis. $H_{\alpha\alpha}(q)$ corresponds to the curvature of $\partial\Omega$ in the direction e_α. By definition, the mean curvature of $\partial\Omega$ at q is given by the trace of this matrix, namely

$$H_{\partial\Omega}(q) = \sum_{\alpha=1}^{n-1} H_{\alpha\alpha}(q).$$

In order to state our result we need to consider the mean of the curvatures in the directions of T_qK and N_qK, namely the numbers $\sum_{a=1}^{k} H_{aa}(q)$ and $\sum_{j=k+1}^{n-1} H_{jj}(q)$.

Theorem 1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, let K be a k-dimensional non degenerate minimal sub-manifold of $\partial\Omega$. Assume that $n-k\geq 7$ and that the mean of the curvatures in the directions of T_qK is negative, namely,*

$$\sum_{a=1}^{k} H_{aa}(q) < 0 \quad \text{for all} \quad q \in K.$$

Then, for a sequence $\varepsilon = \varepsilon_j \longrightarrow 0$, Problem (1.3) has a positive solution u_ε concentrating along K as $\varepsilon \to 0$, in the sense that

$$|\nabla u_\varepsilon|^2 \rightharpoonup S_{n-k}^{\frac{n-k}{2}} \delta_K \quad \text{as} \quad \varepsilon \to 0$$

where δ_K stands for the Dirac measure supported on K and S_{n-k} is an explicit positive constant.

The condition $n-k \geq 7$ appears also in many previous works like [17], it is a technical condition that seems essential for our method to work (see proof of Proposition 3.1 for some comments about this fact) but we believe the phenomenon described should also be true for lower co-dimensions. We also point out that the resonance phenomenon has already been found in the analysis of higher dimensional concentration in other elliptic boundary value problems, in particular for Neumann singular perturbation problem in [24, 27, 28, 29] and nonlinear Schrödinger equations on compact Riemannian manifolds without boundary or in $\mathbb{R}^N$, see [14], [26].

The solution predicted in Theorem 1 can be described as follows: points $x \in \mathbb{R}^n$ near K, can be parametrized as

$$x = q + z, \quad \text{for} \quad q \in K, \quad |z| = \text{dist}\,(x, K).$$

At main order our solution will look like

$$u_\varepsilon(x) \sim \mu_\varepsilon^{-\frac{n-2}{2}}(q) w_{n-k}\left(\frac{x - d_\varepsilon(q)}{\mu_\varepsilon(q)}\right), \tag{1.7}$$

as $\varepsilon \to 0^+$, where w_{n-k} is the *standard bubble* in dimension $n-k$,

$$w_{n-k}(x) = \left(\frac{c_{n-k}}{1+|x|^2}\right)^{\frac{n-k-2}{2}}, \qquad c_{n-k} = ((n-k)(n-k-2))^{\frac{1}{n-k-2}}, \tag{1.8}$$

a radial solution of the corresponding limit problem in $\mathbb{R}^{n-k}$

$$\Delta w + w^{\frac{n-k+2}{n-k-2}} = 0 \quad \text{in } \mathbb{R}^{n-k}. \tag{1.9}$$

In (1.7), $\mu_\varepsilon(q)$ is a strictly positive scalar function that takes into account the invariance of (1.9) under scaling, while $d_\varepsilon(q)$ is a vector function, with values in $\mathbb{R}^{n-k}$, that describes the deviation of the center of the bubble in (1.7) from the manifold K.

The first main ingredient in proving our main theorem is the construction of a very accurate approximate solution in powers of ε and $\rho = \varepsilon^{\frac{N-1}{N-2}}$, in a neighborhood of the scaled sub-manifold $K_\rho = \rho^{-1}K$. It is worth mentioning that concentration at higher dimensional sets for some related problem with Neumann boundary conditions or on manifolds has been extensively studied in the last decade, see [11, 15, 18, 19, 24, 26] and some references therein. In most of the above mentioned problems the profile has an exponential decay which is crucial in the construction of very accurate approximate solutions via an iterative scheme of Picard's type. Here instead the profile (1.8) has a polynomial decay and henceforth much more refined estimates are needed to perform again an iterative procedure to improve the approximation. Another issue is that the profile $U := w_{n-k}$ copied and translated along K, as described in (1.7), does not satisfy zero Dirichlet boundary conditions. Hence one needs to introduce a function $\bar U$, see (3.21) for its definition, to adjust the boundary conditions, and take $U - \bar U$ to be the first approximation. A third observation here is that since the limit problem is critical for dimension $n-k$, the linearized operator have a nontrivial kernel due to invariance of the equation under translations and dilations. This amounts to define some parameter functions μ_ε and some smooth normal sections d_ε to guarantee the solvability of some projected problems. The condition $\sum_{a=1}^k H_{aa}(q) < 0 \quad \text{for all} \quad q \in K$, that appears in the main Theorem 1 is in fact imposed to guarantee the positivity of the main term of the dilation parameter μ_ε, see formula (3.37) below. A more subtle issue we have to take care is the fact that the $(n-k)$-dimensional profile is an unstable solution to (1.9). Indeed w_{n-k} is a Mountain-pass type solution (of Morse index one). The linearized operator about this profile has one negative eigenvalue and as the concentration parameter ε becomes smaller and smaller, this negative eigenvalue generates more and more unstable directions. This is the origin of a resonance phenomena and the reason why our result is valid only for a sequence $\varepsilon = \varepsilon_j \to 0^+$. The Morse index of our solutions diverges as $\varepsilon \to 0$.

Once a very accurate approximate solution is constructed we can built the desired solution by linearizing the main equation around this approximation. The associated linear operator turns out to be invertible with inverse controlled in a suitable norm by certain large negative power of ε, provided that ε remains away from certain critical values where resonance occurs. The interplay of the size of the error and that of the inverse of the linearization then makes it possible a fixed point scheme.

The paper is organized as follows: We first introduce some notations and conventions and we expand the coefficients of the metric near K using geodesic normal coordinates (Fermi coordinates). We then expand the Laplace-Beltrami operator. Section 3 will be mainly devoted to the construction of a local approximate solution. To perform this construction we need a solvability theory and a-priori estimates for a certain linear operator, which is developed in Section 5. In Section 4 we first define a global approximate solution, so that the solution to our problem can be written as the sum of this global approximation plus a remaining "small" term. Then, we prove our main Theorem. To solve such problem, we need to understand the invertibility properties of another linear operator. The Appendix in Section 6 is devoted to prove some technical facts. For brevity, most of the arguments that has been already used in some previous works will be omitted here, referring the reader to precise references.

2. Setting up the problem in geodesic normal coordinates

In this section we first introduce Fermi coordinates near a k-dimensional sub-manifold of $\partial\Omega \subset \mathbb{R}^n$ (with $n = N + k$) and we expand the coefficients of the metric in these coordinates. We will omit details here referring to [15, 18, 24]. We then express our main equation in these Fermi coordinates.

2.1. **Notation and conventions.** Dealing with coordinates, Greek letters like $\alpha, \beta, \dots,$ will denote indices varying between 1 and $n-1$, while capital letters like $A, B, \dots$ will vary between 1 and n; Roman letters like a or b will run from 1 to k, while indices like $i, j, \dots$ will run between 1 and $N-1 := n-k-1$. $\xi_1, \dots, \xi_{N-1}, \xi_N$ will denote coordinates in $\mathbb{R}^N = \mathbb{R}^{n-k}$, and they will also be written as $\bar{\xi} = (\xi_1, \dots, \xi_{N-1})$, $\xi = (\bar{\xi}, \xi_N)$. The manifold K will be parameterized with coordinates $y = (y_1, \dots, y_k)$. Its dilation $K_\rho := \frac{1}{\rho} K$ will be parameterized by coordinates $z = (z_1, \dots, z_k)$ related to the y's simply by $y = \rho z$, where $\rho = \varepsilon^{\frac{N-1}{N-2}}$. Derivatives with respect to the variables y, z or ξ will be denoted by ∂_y, ∂_z, ∂_ξ, and for brevity sometimes we might use the symbols ∂_a, $\partial_{\overline{a}}$ and ∂_i for ∂_{y_a}, ∂_{z_a} and ∂_{ξ_i} respectively.

2.2. **Expansion of the metric in local coordinates.** Let K be a k-dimensional sub-manifold of $(\partial\Omega, \overline{g})$ $(1 \le k \le N-1)$, where $\overline{g}$ is the induced metric on $\partial\Omega$ of the standard metric in $\mathbb{R}^n$. We choose along K a local orthonormal frame field $((E_a)_{a=1,\cdots k}, (E_i)_{i=1,\cdots,N-1})$ which is oriented. At points of K, we have the natural splitting $T\partial\Omega = TK \oplus NK$ where TK is the tangent space to K and NK represents the normal bundle, which are spanned respectively by $(E_a)_a$ and $(E_j)_j$.

We denote by ∇ the connection induced by the metric $\overline{g}$ and by ∇^N the corresponding normal connection on the normal bundle. Given $q \in K$, we use some geodesic coordinates y centered at q. We also assume that at q the normal vectors $(E_i)_i$, $i = 1, \dots, n$, are transported in a parallel way (with respect to ∇^N) through geodesics from q, so in particular

$$\overline{g}\left(\nabla_{E_a} E_j \,, E_i\right) = 0 \quad \text{at } q, \qquad i, j = 1, \dots, n, a = 1, \dots, k. \tag{2.1}$$

In a neighborhood of q in K, we consider normal geodesic coordinates

$$f(y) := \exp_q^K(y_a\, E_a), \qquad y := (y_1, \dots, y_k),$$

where $\exp^K$ is the exponential map on K and summation over repeated indices is understood. This yields the coordinate vector fields $X_a := f_*(\partial_{y_a})$. We extend the E_i along each $\gamma_E(s)$ so that they are parallel with respect to the induced connection on the normal bundle NK. This yields an orthonormal frame field X_i for NK in a neighborhood of q in K which satisfies

$$\nabla_{X_a} X_i\big|_q \in T_q K.$$

A coordinate system in a neighborhood of q in $\partial\Omega$ is now defined by

$$F(y, \bar{x}) := \exp^{\partial\Omega}_{f(y)}(x_i\, X_i), \qquad (y, \bar{x}) := (y_1, \dots, y_k, x_1, \dots, x_{N-1}), \tag{2.2}$$

with corresponding coordinate vector fields

$$X_i := F_*(\partial_{x_i}) \qquad \text{and} \qquad X_a := F_*(\partial_{y_a}).$$

By our choice of coordinates, on K the metric $\overline{g}$ splits in the following way

$$\overline{g}(q) = \overline{g}_{ab}(q)\, dy_a \otimes dy_b + \overline{g}_{ij}(q)\, dx_i \otimes dx_j, \qquad q \in K. \tag{2.3}$$

We denote by $\Gamma_a^b(\cdot)$ the 1-forms defined on the normal bundle, NK, of K by the formula

$$\overline{g}_{bc}\Gamma^c_{ai} := \overline{g}_{bc}\Gamma^c_a(X_i) = \overline{g}(\nabla_{X_a} X_b, X_i) \quad \text{at } q = f(y). \tag{2.4}$$

Note that K is minimal if and only if $\sum_{a=1}^k \Gamma^a_a(E_i) = 0$ for any $i = 1, \dots, N-1$.

Define $q = f(y) = F(y, 0) \in K$ and let $(\tilde{g}_{ab}(y))$ be the induced metric on K. When we consider the metric coefficients in a neighborhood of K, we obtain a deviation from formula (2.3):

$$\begin{aligned} \overline{g}_{ij} &= \delta_{ij} + \tfrac{1}{3}\, R_{istj}\, x_s\, x_t \,+\, \mathcal{O}(|x|^3); \quad \overline{g}_{aj} = \mathcal{O}(|x|^2); \\ \overline{g}_{ab} &= \tilde{g}_{ab} - \left\{ \tilde{g}_{ac}\, \Gamma^c_{bi} + \tilde{g}_{bc}\, \Gamma^c_{ai} \right\} x_i + \left[R_{sabl} + \tilde{g}_{cd} \Gamma^c_{as}\, \Gamma^d_{bl} \right] x_s x_l + \mathcal{O}(|x|^3). \end{aligned}$$

Here $a = 1, \dots, k$ and any $i, j = 1, \dots, N-1$, and $R_{\alpha\beta\gamma\delta}$ the components of the curvature tensor with lowered indices, which are obtained by means of the usual ones $R^\sigma_{\beta\gamma\delta}$ by

$$R_{\alpha\beta\gamma\delta} = \overline{g}_{\alpha\sigma}\, R^\sigma_{\beta\gamma\delta}. \tag{2.5}$$

The proof of these facts can be found in Lemma 2.1 in [15], see also [9, 25, 30].

Next we introduce a parametrization of a neighborhood in Ω of $q \in \partial\Omega$ through the map Υ given by

$$\Upsilon(y,x) = F(y,\bar{x}) + x_N \nu(y,\bar{x}), \qquad x = (\bar{x}, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}, \tag{2.6}$$

where F is the parametrization introduced in (2.2) and $\nu(y,\bar{x})$ is the inner unit normal to $\partial\Omega$ at $F(y,\bar{x})$. We have

$$\frac{\partial \Upsilon}{\partial y_a} = \frac{\partial F}{\partial y_a}(y,\bar{x}) + x_N \frac{\partial \nu}{\partial y_a}(y,\bar{x}); \qquad \frac{\partial \Upsilon}{\partial x_i} = \frac{\partial F}{\partial x_i}(y,\bar{x}) + x_N \frac{\partial \nu}{\partial x_i}(y,\bar{x}).$$

Let us define the tensor matrix H by

$$d\nu_x[v] = -H(x)[v]. \tag{2.7}$$

We thus find

$$\frac{\partial \Upsilon}{\partial y_a} = [Id - x_N H(y,\bar{x})] \frac{\partial F}{\partial y_a}(y,\bar{x}); \quad \text{and} \quad \frac{\partial \Upsilon}{\partial x_i} = [Id - x_N H(y,\bar{x})] \frac{\partial F}{\partial x_i}(y,\bar{x}). \tag{2.8}$$

Differentiating Υ with respect to x_N we also get $\frac{\partial \Upsilon}{\partial x_N} = \nu(y,\bar{x})$.

Hence, letting $g_{\alpha\beta}$ be the coefficients of the flat metric g of $\mathbb{R}^{N+k}$ in the coordinates $(y,\bar{x},x_N)$, with easy computations we deduce for $\tilde{y} = (y,\bar{x})$ that

$$g_{\alpha\beta}(\tilde{y},x_N) = \overline{g}_{\alpha\beta}(\tilde{y}) - x_N \left(H_{\alpha\delta}\overline{g}_{\delta\beta} + H_{\beta\delta}\overline{g}_{\delta\alpha}\right)(\tilde{y}) + x_N^2 H_{\alpha\delta}H_{\sigma\beta}\overline{g}_{\delta\sigma}(\tilde{y}); \quad g_{\alpha N} \equiv 0; \qquad g_{NN} \equiv 1.$$

In the above expressions, with α and β we denote any index of the form $a = 1,\dots,k$ or $i = 1,\dots,N-1$.

For the metric g in the above coordinates $(y,\bar{x},x_N)$ we have the expansions

$$\begin{aligned}
g_{ij} =& \delta_{ij} - 2x_N H_{ij} + \frac{1}{3} R_{istj}\, x_s\, x_t + x_N^2 (H^2)_{ij} + \mathcal{O}((|x|^3), \quad 1 \le i,j \le N-1;\\
g_{aj} =& -x_N \left(H_{aj} + \tilde{g}_{ac} H_{cj} \right) + \mathcal{O}(|x|^2), \quad 1 \le a \le k,\ 1 \le j \le N-1;\\
g_{ab} =& \tilde{g}_{ab} - \left\{ \tilde{g}_{ac}\, \Gamma_{bi}^c + \tilde{g}_{bc}\, \Gamma_{ai}^c \right\} x_i - x_N \left\{ H_{ac}\, \tilde{g}_{bc} + H_{bc}\, \tilde{g}_{ac} \right\}\\
&+ \left[R_{sabl} + \tilde{g}_{cd} \Gamma_{as}^c\, \Gamma_{dl}^b \right] x_s x_l + x_N^2 (H^2)_{ab}\\
&+ x_N\, x_k \left[H_{ac} \{ \tilde{g}_{bf} \Gamma_{ck}^f + \tilde{g}_{cf} \Gamma_{bk}^f \} + H_{bc} \{ \tilde{g}_{af} \Gamma_{ck}^f + \tilde{g}_{cf} \Gamma_{ak}^f \} \right] + \mathcal{O}(|x|^3),\ 1 \le a,b \le k;\\
g_{aN} \equiv& 0, \quad a = 1,\dots,k; \qquad g_{iN} \equiv 0, \quad i = 1,\dots,N-1; \qquad g_{NN} \equiv 1.
\end{aligned}$$

In the above expressions $H_{\alpha\beta}$ denotes the components of the matrix tensor H defined in (2.7), R_{istj} are the components of the curvature tensor as defined in (2.5), Γ_{ai}^b are defined in (2.4) and $\tilde{g}_{ab}$ is the induced metric on K.

Once we have the expression of the metric, it is a matter of computation to derive the Laplace Beltrami operator. We shall do that in expanded and translated variables.

Let $(y,x) \in \mathbb{R}^{k+N}$ be the local coordinates along K introduced in (2.6). We define $\rho = \varepsilon^{\frac{N-1}{N-2}}$ and we let μ_ε be a positive smooth function defined on K and $d_{1,\varepsilon},\dots,d_{N,\varepsilon} : K \longrightarrow \mathbb{R}$ be smooth functions. We next introduce new functions $\tilde{\mu}_\varepsilon$ and $\tilde{d}_{\ell,\varepsilon}$ so that

$$\tilde{\mu}_\varepsilon = \rho\mu_\varepsilon, \quad \text{and} \quad \tilde{d}_\varepsilon = (\varepsilon^2 \bar{d}_\varepsilon, \tilde{d}_{N,\varepsilon}), \quad \text{with} \quad \bar{d}_\varepsilon = (d_{1,\varepsilon},\dots,d_{N-1,\varepsilon}), \quad \tilde{d}_{N,\varepsilon} = \varepsilon d_{N,\varepsilon}. \tag{2.9}$$

We next introduce the following change of variables $z = \frac{y}{\rho} \in K_\rho := \frac{1}{\rho} K$ and $\xi = \frac{x - \tilde{d}_\varepsilon}{\tilde{\mu}_\varepsilon} \in \mathbb{R}^N$ and as before we write $\xi = (\bar{\xi}, \xi_N)$ with

$$\bar{\xi} = \frac{\overline{x} - \varepsilon^2 \bar{d}_\varepsilon}{\rho\mu_\varepsilon}, \quad \xi_N = \frac{x_N - \varepsilon d_{N,\varepsilon}}{\rho\mu_\varepsilon}. \tag{2.10}$$

We now define the function v by

$$u(z,\overline{x},x_N)=(1+\alpha_\varepsilon)\,\tilde{\mu}_\varepsilon^{-\frac{N-2}{2}}\,v\,(z,\,\xi)\,. \tag{2.11}$$

In (2.11), α_ε is some parameter which has to be chosen so that

$$\Delta((1+\alpha_\varepsilon)U)+\rho^{\frac{N-2}{2}\varepsilon}\,((1+\alpha_\varepsilon)U)^{p-\varepsilon}=0\quad\text{in }\mathbb{R}^N$$

where U is standard bubble in $\mathbb{R}^N$ defined in (1.4) ($U=w_N$). This parameter can be computed explicitly as

$$\alpha_\varepsilon=\rho^{\frac{(N-2)^2}{8-2\varepsilon(N-2)}\varepsilon}-1.$$

Let us mention here that with the above change of variables the functions $\tilde{\mu}_\varepsilon$ and $\tilde{d}_\varepsilon$ depend slowly on the variable z. To emphasize the dependence of the above change of variables on μ_ε and $d_\varepsilon=(\bar{d}_\varepsilon,d_{N,\varepsilon})$, we will use the notation

$$u=\mathcal{T}_{\mu_\varepsilon,d_\varepsilon}(v)\qquad\Longleftrightarrow u\quad\text{and}\quad v\quad\text{satisfy (2.11)}. \tag{2.12}$$

We assume now that the functions μ_ε and d_ε are uniformly bounded, as $\varepsilon\to0$, on K. Since the original variables $(y,\bar{x},x_N)\in\mathbb{R}^k\times\mathbb{R}^{N-1}\times\mathbb{R}_+$ are local coordinates along K, we let the variables (z,ξ) vary in the set $\mathcal{D}$ defined by

$$\mathcal{D}=\left\{(z,\bar{\xi},\xi_N)\,:\,\rho z\in K,\quad|\bar{\xi}|<\frac{\delta}{\rho},\quad-\frac{\tilde{d}_{N,\varepsilon}}{\tilde{\mu}_\varepsilon}<\xi_N<\frac{\delta}{\rho}\right\} \tag{2.13}$$

for some fixed positive number δ. We will also use the notation $\mathcal{D}=K_\rho\times\hat{\mathcal{D}}$, where

$$\hat{\mathcal{D}}=\left\{(\bar{\xi},\xi_N)\,:\,|\bar{\xi}|<\frac{\delta}{\rho},\quad-\frac{\tilde{d}_{N,\varepsilon}}{\tilde{\mu}_\varepsilon}<\xi_N<\frac{\delta}{\rho}\right\}.$$

Using the expansions of the metric we can expand the Laplace Beltrami operator in the new variables (z,ξ) in terms of parameter functions $\tilde{\mu}_\varepsilon(y)$ and $\tilde{d}_\varepsilon(y)$. This is the content of next Lemma, whose proof can be seen in [15, Lemma 3.3].

Lemma 2.1. *Given the change of variables defined in (2.11), the following expansion for the Laplace Beltrami operator holds true*

$$(1+\alpha_\varepsilon)^{-1}\mu_\varepsilon^{\frac{N+2}{2}}\Delta u=\mathcal{A}_{\mu_\varepsilon,d_\varepsilon}(v):=\mu_\varepsilon^2\Delta_{K_\rho}v+\Delta_\xi v+\sum_{\ell=0}^{5}\mathcal{A}_\ell v+B(v). \tag{2.14}$$

Above, the expression $\mathcal{A}_k$ denotes the following differential operators

$$\begin{aligned}\mathcal{A}_0v=&\tilde{\mu}_\varepsilon D_{\bar{\xi}}v\,[\Delta_K\tilde{d}_\varepsilon]-\tilde{\mu}_\varepsilon\,\Delta_K\tilde{\mu}_\varepsilon\,(\gamma v+D_\xi v\,[\xi])\\&+\,|\nabla_K\tilde{\mu}_\varepsilon|^2\left[D_{\xi\xi}v\,[\xi]^2+2(1+\gamma)D_\xi v[\xi]+\gamma(1+\gamma)v\right]\\&-\nabla_K\tilde{\mu}_\varepsilon\cdot\left\{2D_{\bar{\xi}\bar{\xi}}v[\bar{\xi}]+ND_{\bar{\xi}}v\right\}[\nabla_K\tilde{d}_\varepsilon]+D_{\bar{\xi}\bar{\xi}}v\,[\nabla_K\tilde{d}_\varepsilon]^2\\&-2\,\tilde{\mu}_\varepsilon\,\tilde{g}^{ab}\left[D_\xi(\frac{1}{\rho}\partial_{\bar{a}}v)[\partial_b\tilde{\mu}_\varepsilon\xi]+D_\xi(\frac{1}{\rho}\partial_{\bar{a}}v)[\partial_b\tilde{d}_\varepsilon]+\gamma\partial_a\tilde{\mu}_\varepsilon\,(\frac{1}{\rho}\partial_{\bar{b}}v)\right],\end{aligned}$$

where we have set $\gamma=\frac{N-2}{2}$,

$$\begin{aligned}\mathcal{A}_1\,v=\sum_{i,j}\Big[&2(\tilde{\mu}_\varepsilon\xi_N+\tilde{d}_{N,\varepsilon})H_{ij}-\frac{1}{3}\sum_{m,l}R_{mijl}(\tilde{\mu}_\varepsilon\xi_m+\tilde{d}_{m,\varepsilon})(\tilde{\mu}_\varepsilon\xi_l+\tilde{d}_{l,\varepsilon})\\&+(\tilde{\mu}_\varepsilon\xi_N+\tilde{d}_{N,\varepsilon})\,Q(H)_{ij}+(\tilde{\mu}_\varepsilon\xi_N+\tilde{d}_{N,\varepsilon})\sum_l\mathfrak{D}^{ij}_{Nl}\,(\tilde{\mu}_\varepsilon\xi_l+\tilde{d}_{l,\varepsilon}x)\Big]\partial^2_{ij}v,\end{aligned}$$

where the function $Q(H)_{ij}$ is defined as

$$Q(H)_{ij} = 3x_N^2\, H_{ik}\, H_{kj} + x_N^2 \left(2\, H_{ia}\, H_{aj} + \tilde{g}^{ab}\, H_{ia}\, H_{bj} \right),$$

and the functions $\mathfrak{D}_{Nk}^{ij}$ are smooth functions of the variable $z = \frac{y}{\rho}$ and uniformly bounded. Furthermore,

$$\mathcal{A}_2 v = \tilde{\mu}_\varepsilon \sum_j \left[\sum_s \frac{2}{3} R_{mssj} + \sum_{m,a,b} \left(\tilde{g}_\varepsilon^{ab}\, R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a \right) \right] (\tilde{\mu}_\varepsilon \xi_m + \tilde{d}_{m,\varepsilon}) \partial_j v,$$

and

$$\mathcal{A}_3 v \ = -\tilde{\mu}_\varepsilon \Big[\mathrm{tr}(H) + (\tilde{\mu}_\varepsilon + \tilde{d}_{N,\varepsilon}) \mathrm{tr}(H^2) \Big] \partial_N v.$$

Moreover

$$\mathcal{A}_4 v = 2(\tilde{\mu}_\varepsilon + \tilde{d}_{N,\varepsilon})(H_{aj} + \tilde{g}^{ac} H_{cj}) \left(\frac{\tilde{\mu}_\varepsilon}{\rho} \partial^2_{\bar{a}j} v - \partial_a \tilde{\mu}_\varepsilon D_\xi(\partial_j v) - D_\xi(\partial_j v)[\partial_a \tilde{d}_\varepsilon] + (1+\gamma) \partial_a \tilde{\mu}_\varepsilon \partial_j v \right)$$

and

$$\mathcal{A}_5 v = \left(\sum_{a,j} \mathfrak{D}_j^a [\tilde{\mu}_\varepsilon \xi_j + \tilde{d}_{j,\varepsilon}] + (\tilde{\mu}_\varepsilon \xi_N + \tilde{d}_{N,\varepsilon})\, \mathfrak{D}_N^a \right) \left\{ \tilde{\mu}_\varepsilon \left[-D_{\bar{\xi}} v\, [\partial_a \tilde{d}_\varepsilon] + \tilde{\mu}_\varepsilon \partial_{\bar{a}} v - \partial_a \tilde{\mu}_\varepsilon (\gamma v + D_\xi v\, [\xi]) \right] \right\}$$

where $\mathfrak{D}_j^a$ and $\mathfrak{D}_N^a$ are smooth functions of $z = \frac{y}{\rho}$. Finally, the operator $B(v)$ is defined below,

$$\begin{aligned} \mathcal{B}(v) =& O\left((\tilde{\mu}_\varepsilon \bar{\xi} + \bar{\tilde{d}}_\varepsilon)^2 + (\tilde{\mu}_\varepsilon \xi_N + \tilde{d}_{N,\varepsilon})(\tilde{\mu}_\varepsilon \bar{\xi} + \bar{\tilde{d}}_\varepsilon) + (\tilde{\mu}_\varepsilon \xi_N + \tilde{d}_{N,\varepsilon})^2 \right) \times \\ & \times \left(-\frac{N}{2}\, \partial_{\bar{a}} \tilde{\mu}_\varepsilon\, \partial_l v + \frac{\tilde{\mu}_\varepsilon}{\varepsilon}\, \partial^2_{\bar{a}l} v - \partial_{\bar{a}} \tilde{\mu}_\varepsilon \xi_J \partial^2_{lJ} v - \partial_{\bar{a}} \Phi^j \partial^2_{lj} v \right) \\ & + O\left((\tilde{\mu}_\varepsilon \bar{\xi} + \bar{d}_\varepsilon)^3 + (\tilde{\mu}_\varepsilon \xi_N + \tilde{d}_{N,\varepsilon})(\tilde{\mu}_\varepsilon \bar{\xi} + \bar{\tilde{d}}_\varepsilon)^2 \right. \\ & \left. + (\tilde{\mu}_\varepsilon \xi_N + \tilde{d}_{N,\varepsilon})^2 (\tilde{\mu}_\varepsilon \bar{\xi} + \bar{\tilde{d}}_\varepsilon) \tilde{d}_\varepsilon| + (\tilde{\mu}_\varepsilon \xi_N + \tilde{d}_{N,\varepsilon})^3 \right) \partial^2_{ij} v, \end{aligned} \tag{2.15}$$

where $\bar{\tilde{d}}_\varepsilon = \varepsilon^2 \bar{d}_\varepsilon$. We recall that the symbols ∂_a, $\partial_{\bar{a}}$ and ∂_i denote the derivatives with respect to ∂_{y_a}, ∂_{z_a} and ∂_{ξ_i} respectively.

2.3. **Expressing the equation in coordinates.** We recall that we want to find a solution to the problem

$$\Delta u + u_+^{p-\varepsilon} = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.16}$$

where $p = \frac{N+2}{N-2}$ with $N = n - k$.

After performing the change of variables in (2.11), the original equation in u reduces near $K_\rho = \frac{K}{\rho}$ to the following equation in v

$$-\mathcal{A}_{\mu_\varepsilon, d_\varepsilon} v - \mu_\varepsilon^{\frac{N-2}{2}\varepsilon} v^{p-\varepsilon} = 0, \tag{2.17}$$

where $\mathcal{A}_{\mu_\varepsilon, d_\varepsilon}$ is defined in (2.14) and $p = \frac{N+2}{N-2}$. We denote by $\mathcal{S}_\varepsilon$ the operator given by (2.17), namely

$$\mathcal{S}_\varepsilon(v) := -\mathcal{A}_{\mu_\varepsilon, d_\varepsilon} v - \mu_\varepsilon^{\frac{N-2}{2}\varepsilon} v^{p-\varepsilon}. \tag{2.18}$$

Recalling the definitions $\tilde{\mu}_\varepsilon = \rho \mu_\varepsilon$, $\tilde{d}_\varepsilon = (\bar{\tilde{d}}_\varepsilon, \tilde{d}_{N,\varepsilon}) = (\varepsilon^2 \bar{d}_\varepsilon, \varepsilon d_{N,\varepsilon})$ with $\bar{d}_\varepsilon = (d_{1,\varepsilon}, \ldots, d_{N-1,\varepsilon})$ in Lemma 2.1, we get the following result which gives the expansion of $\mathcal{S}_\varepsilon(v)$ in powers of ε, ρ and in terms of the real function μ_ε, $d_{N,\varepsilon}$ and the normal section $\bar{d}_\varepsilon$.

Lemma 2.2. *It holds that*

$$
\begin{aligned}
\mathcal{S}_\varepsilon(v) = &- \mu_\varepsilon^2 \Delta_{K_\rho} v - \Delta_\xi v - \mu_\varepsilon^{\frac{N-2}{2}\varepsilon} v^{p-\varepsilon} - \varepsilon\, 2 d_{N,\varepsilon} H_{ij} \partial_{ij} v \\
&- \ \rho\,\{2\mu_\varepsilon \xi_N H_{ij}\partial_{ij} v - \mu_\varepsilon tr(H) \partial_N v\} \\
&- \varepsilon^2\, \mathcal{S}_1(v) - \varepsilon\rho\, \mathcal{S}_2(v) - \rho^2 \mathcal{S}_3(v) - \varepsilon^3 \mathcal{S}_4(v) - \varepsilon^2\rho \mathcal{S}_5(v) - \varepsilon^4 \mathcal{S}_6(v) - B(v),
\end{aligned}
\tag{2.19}
$$

where the terms $\mathcal{S}_j(v)$ are given by

$$
\mathcal{S}_1(v) = |\nabla_K d_{N,\varepsilon}|^2 \partial^2_{NN} v + d^2_{N,\varepsilon} Q(H)_{ij} \partial^2_{ij} v - 2 d_{N,\varepsilon}(H_{aj} + \tilde g^{ac} H_{cj})[\partial_a d_{N,\varepsilon} \partial^2_{Nj} v - \frac{1}{\varepsilon}\mu_\varepsilon \partial^2_{\bar a j} v],
$$

$$
\begin{aligned}
\mathcal{S}_2(v) = &-\mu_\varepsilon \partial_N v[\triangle_K d_{N,\varepsilon}] + 2(1+\gamma)\nabla_K \mu_\varepsilon \partial_N v[\nabla_K d_{N,\varepsilon}] + 2\nabla_K \mu_\varepsilon \partial^2_{\xi\xi_N} v[\xi, \nabla_K d_{N,\varepsilon}] \\
&-2\mu_\varepsilon \tilde g^{ab} \frac{1}{\rho} \partial^2_{N\bar a} v \partial_b d_{N,\varepsilon} + 2\mu_\varepsilon d_{N,\varepsilon} \xi_N Q(H)_{ij} \partial^2_{ij} v - \mu_\varepsilon d_{N,\varepsilon} tr(H^2) \partial_N v \\
&-2(H_{aj} + \tilde g^{ac} H_{cj}) \Big[\mu_\varepsilon \xi_N \partial_a d_{N,\varepsilon} \partial^2_{Nj} v + (1+\gamma) d_{N,\varepsilon} \partial_a \mu_\varepsilon \partial_j v + \frac{1}{\varepsilon}\mu_\varepsilon^2 \xi_N \partial^2_{\bar a j} v \Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_3(v) = &\ \mu_\varepsilon \triangle_K \mu_\varepsilon [\gamma v + D_\xi v[\xi]] - 2\gamma \mu_\varepsilon \nabla_K \mu_\varepsilon \nabla_{K_\rho} v - 2\mu_\varepsilon \tilde g^{ab} \partial_b \mu_\varepsilon D_\xi (\frac{\partial_{\bar a} v}{\rho})[\xi] \\
&+ |\nabla_K \mu_\varepsilon|^2 \left[\gamma(\gamma+1) v + 2(\gamma+1) D_\xi v[\xi] + D^2_{\xi\xi} v[\xi]^2 \right] \\
&- \frac{1}{3} R_{islj} \mu_\varepsilon^2 \xi_s \xi_l \partial^2_{ij} v + \mu_\varepsilon^2 \xi_N^2 Q(H)_{ij} \partial^2_{ij} v + \mu_\varepsilon^2 \xi_N \mathfrak{D}^{ij}_{Nl} \xi_l \partial^2_{ij} v \\
&+ \mu_\varepsilon^2 [\frac{2}{3} R_{mllj} + \tilde g^{ab} R_{jabm} - \Gamma^c_{am} \Gamma^a_{cj}] \xi_m \partial_j v - \mu_\varepsilon^2 \xi_N tr(H^2) \partial_N v \\
&-2(H_{aj} + \tilde g^{ac} H_{cj}) [\mu_\varepsilon \xi_N \partial_a \mu_\varepsilon D_\xi(\partial_j v) + (1+\gamma)\xi_N \mu_\varepsilon \partial_a \mu_\varepsilon \partial_j v],
\end{aligned}
$$

$$
\mathcal{S}_4(v) = \partial^2_{jN} v \nabla_K d_{N,\varepsilon} \nabla_K d_j + d_N \mathfrak{D}^{ij}_{Nl} d_{l,\varepsilon} \partial^2_{ij} v - 2 d_{N,\varepsilon}(H_{aj} + \tilde g^{ac} H_{cj}) \partial_a d_l \partial^2_{jl} v,
$$

$$
\begin{aligned}
\mathcal{S}_5(v) = &-\mu_\varepsilon \partial_j v \triangle_K d_{j,\varepsilon} + \gamma(1+\gamma)\nabla_K \mu_\varepsilon \nabla_K d_{j,\varepsilon} \partial_j v + 2\nabla_K \mu_\varepsilon \nabla_K d_j \partial^2_{jl} v \xi_l \\
&-2\mu_\varepsilon \tilde g^{ab} \frac{1}{\rho} \partial^2_{\bar a j} v \partial_b d_{j,\varepsilon} - \frac{1}{3}\mu_\varepsilon R_{mijl}(\xi_m d_{l,\varepsilon} + \xi_l d_{m,\varepsilon}) \partial^2_{ij} v + \mu_\varepsilon \mathfrak{D}^{ij}_{Nl} \xi_N d_{l,\varepsilon} \partial^2_{ij} v \\
&+\mu_\varepsilon [\frac{2}{3} R_{mllj} + \tilde g^{ab} R_{jabm} - \Gamma^c_{am}\Gamma^a_{cj}] d_{m,\varepsilon} \partial_j v - 2\mu_\varepsilon \xi_N (H_{aj} + \tilde g^{ac} H_{cj}) \partial_a d_{l,\varepsilon} \partial^2_{jl} v,
\end{aligned}
$$

$$
\mathcal{S}_6(v) = \nabla_K d_{j,\varepsilon} \nabla_K d_{i,\varepsilon} \partial^2_{ij} v - \frac{1}{3} R_{islj} d_{s,\varepsilon} d_{l,\varepsilon} \partial^2_{ij} v,
$$

where the functions $\mathfrak{D}^{ij}_{Nk}$ are smooth functions of the variable $z = \frac{y}{\rho}$ and uniformly bounded. Finally, the operator $B(v)$ is defined in (2.15).

3. Construction of local approximate solutions

In this section, we will construct very accurate approximate solutions to our problem. The basic tool for this construction is a linear theory we will describe below. We consider the domain $\mathcal{D}$ defined as (2.13) and for functions ϕ defined on $\mathcal{D}$, an operator of the form

$$
L(\phi) := -\Delta_\xi \phi - pU^{p-1}\phi + b_{ij}(\rho z, \xi)\partial_{ij}\phi + b_i(\rho z, \xi)\partial_i \phi \tag{3.1}
$$

where b_{ij}, b_i and c are functions defined in $\mathcal{D}$, which depend smoothly on $y \in K$. Recall that a variable $z \in K_\rho$ has the form $\rho z = y \in K$.

We want to establish a solvability theory and a-priori bounds for the following linear problem

$$
\begin{cases}
L(\phi) = h, & \text{in } \mathcal{D} \\
\phi = 0, & \text{on } \partial\hat{\mathcal{D}} \\
\int_{\hat{\mathcal{D}}} \phi(\rho z, \xi) Z_j(\xi)\, d\xi = 0 \quad \forall z \in K_\rho, \quad j = 0, \dots N+1,
\end{cases}
\tag{3.2}
$$

for a given function $h : \mathcal{D} \to \mathbb{R}$, which depends smoothly on the variable $y \in K$. The functions $Z_j(\xi)$, $j = 1, \dots, N+1$, are

$$Z_j(\xi) = \frac{\partial U}{\partial \xi_j}, \quad j = 1, \dots, N, \quad Z_{N+1}(\xi) = \xi \cdot \nabla U(\xi) + \frac{N-2}{2} U(\xi). \tag{3.3}$$

It is well known (see for instance [5]) that these functions are the only bounded solutions to the linearized equation around U of problem (1.5) in $\mathbb{R}^N$

$$-\Delta \phi - pU^{p-1}\phi = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Moreover, Z_0 is the first eigenfunction (normalized to have L^2-norm equal to 1) corresponding to the first eigenvalue $\lambda_1 > 0$ $L^2(\mathbb{R}^N)$ of the problem

$$\Delta_\xi \phi + pU(\xi)^{p-1}\phi = \lambda \phi \quad \text{in} \quad \mathbb{R}^N. \tag{3.4}$$

Observe that this eigenfunction decays exponentially at infinity like $\xi \longmapsto |\xi|^{-\frac{N-1}{2}} e^{-\sqrt{\lambda_1}|\xi|}$.

In order to solve the above linear problem, we define the following norms. Let $\delta > 0$ be a positive, small fixed number. Let r be an integer. For a function w defined in $\mathcal{D} = K_\rho \times \hat{\mathcal{D}}$, we define

$$\|w\|_{\varepsilon,r} := \sup_{(z,\xi)\in K_\rho \times \hat{\mathcal{D}}} \left((1+|\xi|^2)^{\frac{r}{2}} |w(z,\xi)| \right). \tag{3.5}$$

Let $\sigma \in (0,1)$. We then set

$$\|w\|_{\varepsilon,r,\sigma} := \|w\|_{\rho,r} + \sup_{(z,\xi)\in K_\rho \times \hat{\mathcal{D}}} \left((1+|\xi|^2)^{\frac{r+\sigma}{2}} [w]_{\sigma,B(\xi,1)} \right) \tag{3.6}$$

where we have denoted

$$[w]_{\sigma,B(\xi,1)} := \sup_{\xi_1,\xi_2 \in B(\xi,1),\, \xi_1 \neq \xi_2} \frac{|w(z,\xi_2) - w(z,\xi_1)|}{|\xi_1 - \xi_2|^\sigma}. \tag{3.7}$$

We will establish existence and uniform a priori estimates for problem (3.2) in the above norms, provided that appropriate bounds for coefficients hold. We have the validity of the following result.

Proposition 3.1. *Assume that $N \geq 7$, and let r be an integer such that $2 < r < N-2$. Then there exist positive numbers δ, C such that if, for all i, j*

$$\|b_{ij}\|_\infty + \|Db_{ij}\|_\infty + \|(1+|y|)b_i\|_\infty < \delta, \tag{3.8}$$

for all $y = \rho z \in \mathbb{R}^k$. Let $h : K \times \hat{\mathcal{D}} \to \mathbb{R}$ be a function that depends smoothly on the variable $y \in K$, such that $\|h\|_{\varepsilon,r}$ is bounded, uniformly in ε, and

$$\int_{\hat{\mathcal{D}}} h(\varepsilon z, \xi) Z_j(\xi) d\xi = 0 \quad \text{for all } z \in K_\rho, \quad j = 0, 1, \dots, N+1.$$

Then there exists a solution ϕ of problem (3.2) and a constant $C > 0$ such that

$$\|D_\xi^2 \phi\|_{\varepsilon,r,\sigma} + \|D_\xi \phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma} \leq C\|h\|_{\varepsilon,r,\sigma}. \tag{3.9}$$

Furthermore, the function ϕ depends smoothly on the variable ρz, and the following estimates hold true: for any integer l there exists a positive constant C_l such that

$$\|D_y^l \phi\|_{\varepsilon,r-2,\sigma} \leq C_l \left(\sum_{k\leq l} \|D_y^k h\|_{\varepsilon,r,\sigma} \right). \tag{3.10}$$

Proof. The proof is adapted from Proposition 3.1 in [18]. We give here the main ideas for completeness.

First, we prove a priori estimates by dividing the argument into several steps.

Step 1. Let us assume that in problem (3.2) the coefficients b_{ij}, b_i are identically zero. Thus assume that ϕ is a solution to

$$\begin{cases} -\Delta\phi - pw_0^{p-1}\phi = h & \text{in } \mathcal{D} \\ \phi = 0, & \text{on } \partial\hat{\mathcal{D}} \\ \int_{\hat{\mathcal{D}}} \phi(\rho z,\xi) Z_j(\xi)\, d\xi = 0 & \text{for all} \quad z \in K_\rho, \quad j = 0,\dots N+1. \end{cases} \tag{3.11}$$

We claim that there exists $C > 0$ such that

$$\|\phi\|_{\varepsilon,r-2} \le C\|h\|_{\varepsilon,r}. \tag{3.12}$$

By contradiction, assume that there exist sequences $\varepsilon_n \to 0$ (note that $\rho_n = \varepsilon_n^{\frac{N-1}{N-2}} \to 0$), h_n with $\|h_n\|_{\varepsilon_n,r} \to 0$ and solutions ϕ_n to (3.11) with $\|\phi_n\|_{\varepsilon_n,r-2} = 1$.

Let $z_n \in K_{\rho_n}$ and ξ_n be such that $|\phi_n(\rho_n z_n,\xi_n)| = \sup|\phi_n(y,\xi)|$. We may assume that, up to subsequences, $\rho_n z_n \to \bar{y}$ in K. In particular one gets that $|\xi_n| \le C\rho_n^{-1}$ for some positive constant C independent of ε_n.

Let us now assume that there exists a positive constant M such that $|\xi_n| \le M$. In this case, up to subsequences, one gets that $\xi_n \to \xi_0$. Consider the functions $\tilde\phi_n(z,\xi) = \phi_n(z,\xi+\xi_n)$. This is a sequence of uniformly bounded functions that converges uniformly over compact sets of $K \times \hat{\mathcal{D}}$ to a solution $\tilde\phi$ of $-\Delta\tilde\phi - pw_0^{p-1}\tilde\phi = 0$ in $\mathbb{R}^N$. Assuming $N \ge 7$, $2 < r < N-2$ and using the dominated convergence theorem, the orthogonality conditions pass to the limit and we get

$$\int_{\mathbb{R}^N} \tilde\phi(y,\xi) Z_j(\xi)\, d\xi = 0 \qquad \text{for all} \quad y \in K, \qquad \text{for all} \quad j = 0,\dots N+1.$$

These facts imply that $\tilde\phi \equiv 0$, that is a contradiction. We point out here that our condition $N \ge 7$ is needed to guarantee the the integrability of $(1+|\xi|^2)^{-\frac{r-2}{2}} Z_{N+1}(\xi)$.

Assume now that $\lim_{n\to\infty} |\xi_n| = \infty$. Consider the scaled function

$$\tilde\phi_n(z,\xi) = \phi_n(z,|\xi_n|\xi + \xi_n)$$

defined on the set

$$\bar{\mathcal{D}} = \left\{ (z,\bar\xi,\xi_N) : \ |\bar\xi| < \frac{\delta}{\rho_n|\xi_n|} - \frac{\xi_n}{|\xi_n|}, \ -\frac{\varepsilon_n d_{N,\varepsilon_n}}{\rho_n \mu_{\varepsilon_n}|\xi_n|} - \frac{\xi_n}{|\xi_n|} < \xi_N < \frac{\delta}{\rho_n|\xi_n|} - \frac{\xi_n}{|\xi_n|} \right\}.$$

Thus $\tilde\phi_n$ satisfies the equation

$$-\Delta\tilde\phi_n - p\, c_N^{p-1} \frac{|\xi_n|^2}{(1+|\,|\xi_n|\xi+\xi_n|^2)^2}\tilde\phi_n = |\xi_n|^2 h(z,|\xi_n|\xi+\xi_n) \ \text{ in } \bar{\mathcal{D}}.$$

Under our assumptions, we have that $\tilde\phi_n$ is uniformly bounded and it converges locally over compact sets to $\tilde\phi$ solution to $\Delta\tilde\phi = 0$, $\quad |\tilde\phi| \le C|\xi|^{2-r} \quad$ in $\quad \mathbb{R}^N$. Since $2 < r < N$, we conclude that $\tilde\phi \equiv 0$, which is a contradiction. The proof of (3.12) is completed.

Step 2. We shall now show that there exists $C > 0$ such that, if ϕ is a solution to (3.11), then

$$\|D_\xi^2\phi\|_{\varepsilon,r} + \|D_\xi\phi\|_{\varepsilon,r-1} + \|\phi\|_{\varepsilon,r-2} \le C\|h\|_{\varepsilon,r}. \tag{3.13}$$

For $z \in K_\rho$, we have that ϕ solves $-\Delta_\xi\phi = h + pw_0^{p-1}\phi := \tilde h$ in $\mathcal{D}$. From Step 1, we have that $|\tilde h| \le \frac{\|h\|_{\varepsilon,r}}{(1+|\xi|^r)}$. Elliptic estimates give that $|\phi| \le \frac{C}{(1+|\xi|^{r-2})}$.

Let us now fix a point $e \in \mathbb{R}^N$ and a positive number $R > 0$. Perform the change of variables $\tilde\phi(z,t) = R^{r-2}\,\phi(z,Rt+3Re)$, so that

$$\Delta\tilde\phi = R^r\tilde h(z,Rt+3Re) \quad \text{in} \quad |t| \le 1.$$

Elliptic estimates give then that

$$\|D^2\tilde\phi\|_{L^\infty(B(0,1))} + \|D\tilde\phi\|_{L^\infty(B(0,1))} \le C\|R^r\tilde h(z,Rt+3Re)\|_{L^\infty(B(0,2))}.$$

It then follows that

$$\|(1+|\xi|)^r D^2\phi\|_{L^\infty(|\xi|\le\delta\rho^{-1})} \le C\|(1+|\xi|)^r h\|_{L^\infty(|\xi|\le\delta\rho^{-1})}.$$

Arguing in a similar way, one gets the internal weighted estimate for the first derivative of ϕ

$$\|(1+|\xi|)^{r-1} D\phi\|_{L^\infty(|\xi|\le\delta\rho^{-1})} \le C\|(1+|\xi|)^r h\|_{L^\infty(|\xi|\le\delta\rho^{-1})}.$$

By using the representation formula for solution ϕ to the above equation, we see that $|\phi| \le C\rho^{\frac{r-2}{2}}$ in $|\xi| < \delta\rho^{-1}$. Furthermore, elliptic estimates give that in this region $|D\phi| \le C\rho^{\frac{r-1}{2}}$ and $|D^2\phi| \le C\rho^{\frac{r}{2}}$. This concludes the proof of (3.13).

Step 3. We shall now show that there exists $C > 0$ such that, if ϕ is a solution to (3.11), then

$$\|D^2_\xi\phi\|_{\varepsilon,r,\sigma} + \|D_\xi\phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma} \le C\|h\|_{\varepsilon,r,\sigma}. \tag{3.14}$$

Let us first assume we are in the region $|\xi| < \delta\rho^{-1}$, and $z \in K_\rho$. We first claim that from elliptic regularity, we have that if $\|h\|_{\varepsilon,r,\sigma} \le C$ then $\|\phi\|_{\varepsilon,r-2,\sigma} \le C$. Thus, we write that ϕ solves $-\Delta\phi = \tilde h$ in $|\xi| < \delta\rho^{-1}$ where $\|\tilde h\|_{\varepsilon,r,\sigma} \le C$.

Arguing as in the previous step, we fix a point $e \in \mathbb{R}^N$ and a positive number $R > 0$. Perform the change of variables $\tilde\phi(z,t) = R^{r-2}\,\phi(z, Rt + 3Re)$, so that

$$\Delta\tilde\phi = R^r \tilde h(z, Rt + 3Re) \quad \text{in} \quad |t| \le 1.$$

Elliptic estimates give then that $\|D^2\tilde\phi\|_{C^{0,\sigma}(B(0,1))} \le C\|\tilde h\|_{C^{0,\sigma}(B(0,2))}$. This implies that

$$\|D^2_\xi\tilde\phi\|_{L^\infty(B_1)} + [D^2\tilde\phi]_{\sigma,B(0,1)} \le C.$$

In particular, we have for any $z \in K_\rho$, that

$$\sup_{y_1,y_2\in B(0,1)} \frac{|D^2\tilde\phi(z,y_1) - D^2\tilde\phi(z,y_2)|}{|y_1-y_2|^\sigma} \le C.$$

This inequality gets translated in terms of ϕ as

$$R^{r+\sigma} \sup_{\xi_1,\xi_2\in B(\xi,1)} \frac{|D^2\phi(z,\xi_1) - D^2\phi(z,\xi_2)|}{|\xi_1-\xi_2|^\sigma} \le C.$$

In a very similar way, one gets the estimate on $D\phi$. This concludes the proof of (3.14).

Step 4. Differentiating equation (3.11) with respect to the z variable l times and using elliptic regularity estimates, one proves that

$$\|D^l_y\phi\|_{\varepsilon,r-2,\sigma} \le C_l\left(\sum_{k\le l}\|D^k_y h\|_{\varepsilon,r,\sigma}\right) \tag{3.15}$$

for any given integer l.

Step 5. Assume now that the function b_{ij} and b_i in (3.2) are not zero, and assume that ϕ is a solution of problem (3.2), then by (3.14) we obtain

$$\begin{aligned}&\|D^2_\xi\phi\|_{\varepsilon,r,\sigma} + \|D_\xi\phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma}\\ \le& C\|h\|_{\varepsilon,r,\sigma} + C\|b_{ij}\partial_{ij}\phi\|_{\varepsilon,r,\sigma} + C\|b_i\partial_i\phi\|_{\varepsilon,r,\sigma}.\end{aligned}$$

By definition of the norms and from (3.8), we have

$$\|b_{ij}\partial_{ij}\phi\|_{\varepsilon,r,\sigma} + \|b_i\partial_i\phi\|_{\varepsilon,r,\sigma} \le C\delta\left(\|D^2_\xi\phi\|_{\varepsilon,r,\sigma} + \|D_\xi\phi\|_{\varepsilon,r-1,\sigma} + \|\phi\|_{\varepsilon,r-2,\sigma}\right).$$

Therefore, taking $\delta > 0$ small enough, we get (3.9). Also we get (3.10) as a consequence of (3.15).

Next we prove the existence of the solution ϕ to problem (3.11). To this purpose we consider the Hilbert space $\mathcal{H}$ defined as the subspace of functions ψ which are in $H^1(\mathcal{D})$ such that $\psi = 0$ on $\partial\hat{\mathcal{D}}$, and

$$\int_{\hat{\mathcal{D}}} \psi(\rho z, \xi) Z_j(\xi)\, d\xi = 0 \quad \text{for all} \quad z \in K_\varepsilon, \quad j = 0,\dots N+1.$$

Define a bilinear form in $\mathcal{H}$ by $B(\phi,\psi) := \int_{\hat{\mathcal{D}}} \psi L\phi$. Then problem (3.2) gets weakly formulated as that of finding $\phi \in \mathcal{H}$ such that $B(\phi,\psi) = \int_{\hat{\mathcal{D}}} h\psi \quad \forall\ \psi \in \mathcal{H}$. By the Riesz representation theorem, this is equivalent to solving $\phi = T(\phi) + \tilde{h}$ with $\tilde{h} \in \mathcal{H}$ depending linearly on h, and $T : \mathcal{H} \to \mathcal{H}$ being a compact operator. Fredholm's alternative guarantees that there is a unique solution to problem (3.2) for any h provided that

$$\phi = T(\phi) \tag{3.16}$$

has only the zero solution in $\mathcal{H}$. Equation (3.16) is equivalent to problem (3.2) with $h = 0$. If $h = 0$, the estimate in (3.9) implies that $\phi = 0$. This concludes the proof of Proposition 3.1. □

Now we show how we can construct very accurate approximate solutions to Problem (2.17) locally close to K_ρ, using an iterative method that we describe below: let I be an integer. The expanded variables (z, ξ) will be defined as in (2.10) with

$$\mu_\varepsilon(y) = \mu_{0,\varepsilon} + \mu_{1,\varepsilon} + \cdots + \mu_{I,\varepsilon}, \quad y = \rho z \tag{3.17}$$

where $\mu_{0,\varepsilon}, \mu_{1,\varepsilon}, \ldots, \mu_{I,\varepsilon}$ will be smooth functions on K, with $\mu_{0,\varepsilon} = \mu_0 + \varepsilon^{\frac{1}{N-2}} \bar{\mu}_0$, $\mu_0 > 0$ as defined in (3.37). Moreover

$$d_{j,\varepsilon}(y) = d^0_{j,\varepsilon} + d^1_{j,\varepsilon} + \cdots + d^I_{j,\varepsilon}, \quad j = 1, \ldots, N, \tag{3.18}$$

where $d^\ell_{j,\varepsilon}$, $j = 1, \ldots, N$; $\ell = 0, \ldots, I$, will be smooth functions defined along K with values in $\mathbb{R}$, with $d^0_{N,\varepsilon} = d^0_N + \varepsilon^{\frac{1}{N-2}} \bar{d}^0_N$, $d^0_N > 0$ as defined in (3.37). In the (z, ξ) variables, the shape of the approximate solution will be given by

$$v_{I+1,\varepsilon}(z, \bar{\xi}, \xi_N) = \tilde{\omega}_{I+1,\varepsilon} + \tilde{e}_\varepsilon(y) \chi_\varepsilon(\xi) Z_0, \quad y = \rho z \in K, \tag{3.19}$$

with

$$\tilde{\omega}_{I+1,\varepsilon} = U(\xi) - \bar{U}(\xi) + w_{1,\varepsilon}(z, \xi) + \cdots + w_{I+1,\varepsilon}(z, \xi), \quad \xi = (\bar{\xi}, \xi_N) \tag{3.20}$$

where $\bar{U}$ is given by

$$\bar{U}(\xi) = U\left(\bar{\xi}, \xi_N + 2\frac{\tilde{d}_{N,\varepsilon}}{\tilde{\mu}_\varepsilon}\right) = \frac{\alpha_N}{(1 + |\bar{\xi}|^2 + |\xi_N + 2\frac{\tilde{d}_{N,\varepsilon}}{\tilde{\mu}_\varepsilon}|^2)^{\frac{N-2}{2}}}, \quad \alpha_N = (N(N-2))^{\frac{N-2}{4}}, \tag{3.21}$$

and the functions $w_{j,\varepsilon}$'s for $j \geq 1$ are to be determined so that the above function $v_{I+1,\varepsilon}$ satisfies formally

$$\mathcal{S}_\varepsilon(v_{I+1,\varepsilon}) = -\mathcal{A}_{\mu_\varepsilon, d_\varepsilon} v_{I+1,\varepsilon} - \mu_\varepsilon^{\frac{N-2}{2}\varepsilon} v_{I+1,\varepsilon}^{\frac{N+2}{N-2}-\varepsilon} = \mathcal{O}(\varepsilon^{I+2}) \quad \text{in} \quad K_\rho \times \hat{\mathcal{D}}.$$

In the second term in (3.19), Z_0 denotes the first eigenfunction in $L^2(\mathbb{R}^N)$ of the problem

$$\Delta\phi + pU^{p-1}\phi = \lambda\phi \quad \text{in} \quad \mathbb{R}^N, \quad \lambda_1 > 0$$

with $\int Z_0^2 = 1$ and χ_ε is a cut off function defined as follows. Let $\chi = \chi(s)$, for $s \in \mathbb{R}$, with $\chi(s) = 1$ if $s < \hat{\delta}$, $\chi(s) = 0$ if $s > 2\hat{\delta}$, for some fixed $\hat{\delta} > 0$ to be chosen in such a way that $\chi_\varepsilon(\bar{\xi}, -\frac{\tilde{d}_{N,\varepsilon}}{\tilde{\mu}_\varepsilon}) = 0$, where $\chi_\varepsilon(\xi) = \chi(\varepsilon^{\frac{1}{N-2}}|\xi|)$. Observe that the function v_{I+1} satisfies the Dirichlet boundary condition for $\xi_N = -\frac{\tilde{d}_{N,\varepsilon}}{\tilde{\mu}_\varepsilon}$.

Finally, in (3.19) the function $\tilde{e}_\varepsilon(\rho z)$ is defined as follows

$$\tilde{e}_\varepsilon = \varepsilon e_\varepsilon = \varepsilon(e_0 + e_{1,\varepsilon} + \cdots + e_{I,\varepsilon}) \tag{3.22}$$

where $e_{0,\varepsilon} = e_0 + \varepsilon^{\frac{1}{N-2}} \bar{e}_0$, with e_0 is an explicit smooth function, uniformly bounded in ε, whose expression is given in (3.38).

The next proposition shows existence and qualitative properties of the functions μ_ε, d_ε and $v_{I+1,\varepsilon}$ as described above. We prove the following result.

Proposition 3.2. *For any integer $I \in \mathbb{N}$ there exist smooth functions $\mu_\varepsilon : K \to \mathbb{R}$ and $d_{1,\varepsilon}, \ldots, d_{N,\varepsilon} : K \to \mathbb{R}^N$, $e_\varepsilon : K \to \mathbb{R}$, such that*

$$\|\mu_\varepsilon\|_{L^\infty(K)} + \|\partial_a \mu_\varepsilon\|_{L^\infty(K)} + \|\partial^2_a \mu_\varepsilon\|_{L^\infty(K)} \leq C \tag{3.23}$$

$$\|d_{j,\varepsilon}\|_{L^\infty(K)} + \|\partial_a d_{j,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a d_{j,\varepsilon}\|_{L^\infty(K)} \leq C, \quad \textit{for } j = 1, \ldots, N, \tag{3.24}$$

$$\|e_\varepsilon\|_{L^\infty(K)} + \|\partial_a e_\varepsilon\|_{L^\infty(K)} + \|\partial^2_a e_\varepsilon\|_{L^\infty(K)} \leq C \tag{3.25}$$

for some positive constant C, independent of ε. Moreover there exists a positive function $v_{I+1,\varepsilon}$: $K_\rho \times \hat{\mathcal{D}} \to \mathbb{R}$ such that

$$-\mathcal{A}_{\mu_\varepsilon,d_\varepsilon}(v_{I+1,\varepsilon}) - \mu_\varepsilon^{\frac{N-2}{2}\varepsilon} v_{I+1,\varepsilon}^{p-\varepsilon} = \mathcal{E}_{I+1,\varepsilon} \quad in \quad \mathcal{D}$$

$$v_{I+1,\varepsilon} = 0 \quad on \quad \partial\mathcal{D}$$

with

$$\|v_{I+1,\varepsilon} - v_{I,\varepsilon}\|_{\varepsilon,2,\sigma} \leq C\varepsilon^{I+1}, \quad \|\mathcal{E}_{I+1,\varepsilon}\|_{\varepsilon,4,\sigma} \leq C\varepsilon^{I+2}. \tag{3.26}$$

To construct such accurate approximate solutions, we use an iterative scheme of Picard's type. Similar arguments have been used in previous works, but in turns out that in this paper some are more involved. For this reason we give a full detailed construction here. This is the aim of the rest of this section.

Construction of $w_{1,\varepsilon}$, $\mu_{0,\varepsilon}$, $d^0_{N,\varepsilon}$ and $e_{0,\varepsilon}$: For this first step we define

$$v_{1,\varepsilon} = U - \bar{U} + w_{1,\varepsilon} + \varepsilon e_{0,\varepsilon}\chi_\varepsilon(\xi)Z_0$$

with $\mu_\varepsilon = \mu_{0,\varepsilon}$, $\quad d_{N,\varepsilon} = d^0_{N,\varepsilon}$, $\quad$ and $\quad e_\varepsilon = e_{0,\varepsilon}$. Using the expansion of $S_\varepsilon(v_{1,\varepsilon})$ given in Lemma 2.2 with $U = w_N$ is the standard bubble defined in (1.4), we then have, in $\mathcal{D}$,

$$\begin{aligned}
\mathcal{S}_\varepsilon(v_{1,\varepsilon}) =& - \mathcal{A}_{\mu_\varepsilon,d_\varepsilon}\left(U - \bar{U} + w_{1,\varepsilon} + \varepsilon e_{0,\varepsilon}Z_0\right) - \mu_{0,\varepsilon}^{\frac{N-2}{2}\varepsilon}\left(U - \bar{U} + w_{1,\varepsilon} + \varepsilon e_{0,\varepsilon}Z_0\right)^{p-\varepsilon} \\
=& - \Delta_{\mathbb{R}^N} w_{1,\varepsilon} - pU^{p-1}w_{1,\varepsilon} - 2(\varepsilon\, d^0_{N,\varepsilon} + \rho\,\mu_{0,\varepsilon}\xi_N)H_{ij}\partial_{ij}w_{1,\varepsilon} + \rho\mu_{0,\varepsilon}H_{\alpha\alpha}\partial_N w_{1,\varepsilon} \\
& + \bar{U}^p + pU^{p-1}\bar{U} + \varepsilon\left\{U^p\log U - \frac{N-2}{2}U^p\log(\mu_{0,\varepsilon}) - 2d^0_{N,\varepsilon}H_{ij}\partial^2_{ij}U - \lambda_1 e_{0,\varepsilon}Z_0\right\} \\
& - \rho\mu_{0,\varepsilon}\left[2\xi_N H_{ij}\partial^2_{ij}U - H_{\alpha\alpha}\partial_N U\right] + \varepsilon^2\mathcal{E}_{1,\varepsilon} + Q_\varepsilon(w_{1,\varepsilon}) \\
=& \mathcal{L}_\varepsilon w_{1,\varepsilon} + h_{1,\varepsilon} + \varepsilon^2\mathcal{E}_{1,\varepsilon} + Q_\varepsilon(w_{1,\varepsilon}),
\end{aligned}$$

where the operator $\mathcal{L}_\varepsilon$ is given by

$$\mathcal{L}_\varepsilon w_{1,\varepsilon} := -\Delta_{\mathbb{R}^N} w_{1,\varepsilon} - pU^{p-1}w_{1,\varepsilon} - 2(\varepsilon\, d_N + \rho\,\mu_\varepsilon\xi_N)H_{ij}\partial_{ij}w_{1,\varepsilon} + \rho\mu_\varepsilon tr(H)\partial_N w_{1,\varepsilon}. \tag{3.27}$$

The term $h_{1,\varepsilon}$ is defined as follow

$$\begin{aligned}
h_{1,\varepsilon} =& pU^{p-1}\bar{U} + \varepsilon\left\{U^p\log U - \frac{N-2}{2}U^p\log(\mu_{0,\varepsilon}) - 2d^0_{N,\varepsilon}H_{ij}\partial^2_{ij}U - \lambda_1 e_{0,\varepsilon}Z_0\right\} \\
& - \rho\mu_{0,\varepsilon}\left[2\xi_N H_{ij}\partial^2_{ij}U - H_{\alpha\alpha}\partial_N U\right].
\end{aligned} \tag{3.28}$$

The function $\mathcal{E}_{1,\varepsilon}$ is a function which is a sum of functions of the form

$$\begin{aligned}
& f_1(\rho z)\left[f_2(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}, \partial_a\mu_{0,\varepsilon}, \partial_a d^0_{N,\varepsilon}, \partial_e e_{0,\varepsilon}) +\right. \\
& \left.+ o(1)f_3(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}, \partial_a\mu_{0,\varepsilon}, \partial_a d^0_{N,\varepsilon}, \partial_a e_{0,\varepsilon}, \partial^2_{aa}\mu_{0,\varepsilon}, \partial^2_{aa}d^0_{N,\varepsilon}, \partial^2_{aa}e_{0,\varepsilon})\right] f_4(y)
\end{aligned} \tag{3.29}$$

with f_1 a smooth function uniformly bounded in ε, f_2 and f_3 are smooth functions of their arguments, uniformly bounded in ε as $\mu_{0,\varepsilon}$, $d^0_{N,\varepsilon}$ and $e_{0,\varepsilon}$ are uniformly bounded. An important remark is that the function f_3 depends linearly on the argument. Concerning f_4, we have

$$\sup(1 + |\xi|^4)|f_4(y)| < +\infty.$$

The term $Q_\varepsilon(w_{1,\varepsilon})$ is explicitly given by

$$\mu_{0,\varepsilon}^{\frac{N-2}{2}\varepsilon}\left[(U - \bar{U} + w_{1\varepsilon} + \varepsilon e_{0,\varepsilon}Z_0)^{p-\varepsilon} - U^{p\pm\varepsilon} - pU^{p-1\pm\varepsilon}(\bar{U} + w_{1,\varepsilon} + \varepsilon e_{0,\varepsilon}Z_0)\right].$$

We now define $\mu_\varepsilon = \mu_{0,\varepsilon}$, $d_{N,\varepsilon} = d^0_{N,\varepsilon}$, and $e_\varepsilon = e_{0,\varepsilon}$ in such a way that

$$\int_{\hat{\mathcal{D}}} h_{1,\varepsilon}Z_l d\xi = 0 \qquad \text{for all} \quad l = 0, 1, \ldots, N. \tag{3.30}$$

Since $h_{1,\varepsilon}$ is an even function on the variable $\bar\xi$ (due to the fact that U and $\bar U$ are even in $\bar\xi$) since the set $\hat{\mathcal{D}}$ is symmetric in the variable $\bar\xi$, the above condition is automatically satisfied for any $l=1,\ldots,N-1$.

On the other hand, we have (see Section 6 for a proof)

$$\int_{\hat{\mathcal{D}}} h_{1,\varepsilon} Z_{N+1} d\xi = \varepsilon\left[-A_1\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-2} + A_2 + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1} g_{N+1}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)\right](1+o(1)), \tag{3.31}$$

$$\int_{\hat{\mathcal{D}}} h_{1,\varepsilon} Z_{N} d\xi = \varepsilon^{1+\frac{1}{N-2}}\left[A_3\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1} + A_6\mu_{0,\varepsilon} H_{aa} + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N} g_{N}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)\right](1+o(1)), \tag{3.32}$$

and

$$\begin{aligned}\int_{\hat{\mathcal{D}}} h_{1,\varepsilon} Z_0 d\xi \;&=\; \varepsilon\left[A_4\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-2} + A_5 - A_7\log(\mu_{0,\varepsilon}) - \lambda_1 e_{0,\varepsilon} - 2H_{jj} d^0_{N,\varepsilon}\int_{\mathbb{R}^N}\partial^2_{jj} U Z_0 d\xi\right.\\ &\left.+\varepsilon^{\frac{1}{N-2}}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1} g_0\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)\right](1+o(1))\end{aligned} \tag{3.33}$$

where the functions g_i are smooth with $g_i(0)\neq 0$ and A_i are positive constants.

Let $(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}) : K \to\in (0,\infty)\times(0,\infty)\times\mathbb{R}$ be the solution to the following system of nonlinear algebraic equations

$$\begin{cases} -A_1\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-2} + A_2 + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1} g_{N+1}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right) = 0 \\[2ex] A_1\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1} + \frac{A_1A_6}{A_3}\mu_{0,\varepsilon}H_{aa} + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N} g_{N}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right) = 0 \\[2ex] A_4\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-2} + A_5 - A_7\log(\mu_{0,\varepsilon}) - \lambda_1 e_{0,\varepsilon} \\ \quad -2H_{jj}d^0_{N,\varepsilon}\int_{\mathbb{R}^N}\partial^2_{jj}UZ_0d\xi + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1} g_0\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right) = 0\ . \end{cases} \tag{3.34}$$

This solution $(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon})$ exists and has the form

$$\mu_{0,\varepsilon} = \mu_0 + \varepsilon^{\frac{1}{N-2}}\bar\mu_0, \quad d^0_{N,\varepsilon} = d^0_N + \varepsilon^{\frac{1}{N-2}}\bar d^0_N, \quad e_{0,\varepsilon} = e_0 + \varepsilon^{\frac{1}{N-2}}\bar e_0, \tag{3.35}$$

where μ_0, d^0_N and e_0 solve

$$F(\mu_0, d^0_N, e_0) = 0 \tag{3.36}$$

where

$$F(\mu, d_N, e) := \begin{pmatrix} -A_1\left(\frac{\mu}{d_N}\right)^{N-2} + A_2 \\ A_1\left(\frac{\mu}{d_N}\right)^{N-1} + \frac{A_1A_6}{A_3}\mu H_{aa} \\ A_4\left(\frac{\mu}{d_N}\right)^{N-2} + A_5 - A_7\log(\mu) - \lambda_1 e - 2H_{jj}d_N\int_{\mathbb{R}^N}\partial^2_{jj}UZ_0d\xi \end{pmatrix}$$

Explicitly, we get

$$\mu_0 = -\left(\frac{A_2}{A_1}\right)^{\frac{N-1}{N-2}}\frac{A_3}{A_6}\frac{1}{H_{aa}}, \qquad d^0_N = -\frac{A_2}{A_1}\frac{A_3}{A_6}\frac{1}{H_{aa}} \tag{3.37}$$

and

$$e_0 = \frac{1}{\lambda_1}\Big\{ -2d_N^0 H_{jj}\int_{\mathbb{R}^N}\partial^2_{jj} U Z_0 d\xi + \frac{A_2A_4}{A_1} + A_5 - A_7\log(\mu_0)\Big\}. \tag{3.38}$$

Exactly at this point that we need to assume that the mean curvature in the directions of T_qK is negative for any $q\in K$ in order to ensure that μ_0 is positive.

Direct computations give

$$F_0 := \nabla_{\mu,d_N,e}F(\mu_0,d_N^0,e_0) = \begin{pmatrix} -(N-2)A_1\frac{\mu_0^{N-3}}{(d_N^0)^{N-2}} & (N-2)A_1\frac{\mu_0^{N-2}}{(d_N^0)^{N-1}} & 0 \\ (N-2)A_1\frac{\mu_0^{N-2}}{(d_N^0)^{N-1}} & -(N-1)A_1\frac{\mu_0^{N-1}}{(d_N^0)^{N}} & 0 \\ a_{31} & a_{32} & -\lambda_1 \end{pmatrix},$$

where

$$a_{31} = (N-2)A_4\frac{\mu_0^{N-3}}{(d_N^0)^{N-2}} - \frac{A_7}{\mu_0}, \qquad a_{32} = -(N-2)A_4\frac{\mu_0^{N-2}}{(d_N^0)^{N-1}} - 2H_{jj}d_N^0\int_{\mathbb{R}^N}\partial^2_{jj}UZ_0d\xi.$$

Since

$$\det(F_0) = -\lambda_1(N-2)A_1^2\frac{\mu_0^{N-2}}{(d_N^0)^{N-1}}H_{aa} > 0,$$

solving system (3.34) is equivalent to solve a fixed point problem, which is uniquely solvable in the set

$$\Big\{(\bar\mu_0,\bar d_N^0,\bar e_0)\ :\ \|\bar\mu_0\|_\infty\le\delta,\ \|\bar d_N^0\|_\infty\le\delta,\ \|\bar e_0\|_\infty\le\delta\Big\}$$

for some proper small $\delta>0$. Moreover, the smoothness of $\bar\mu_0, \bar d_N^0, \bar e_0$ follows using of the Implicit function Theorem.

For a later purpose we define the following quantities which appeared in the above matrix F_0

$$A := -(N-2)A_1\frac{\mu_0^{N-3}}{(d_N^0)^{N-2}},\quad B = (N-2)A_1\frac{\mu_0^{N-2}}{(d_N^0)^{N-1}},\quad C = -(N-1)A_1\frac{\mu_0^{N-1}}{(d_N^0)^{N}}.$$

An easy computation shows that $AC-B^2>0$.

With the choice for $\mu_{0,\varepsilon}$, $d^0_{N,\varepsilon}$ and $e_{0,\varepsilon}$ in (3.35), the integral of the right hand side in (3.39) against Z_l, $l=0,1,\ldots,N+1$, vanishes on $\hat{\mathcal{D}}$. Furthermore, with this same choice, the linear operator $\mathcal{L}_\varepsilon$ defined in (3.27) satisfies the assumptions of Proposition 3.1. Thus, we define $w_{1,\varepsilon}$ to be solution of the Problem

$$\mathcal{L}_\varepsilon w_{1,\varepsilon} = -h_{1,\varepsilon} \quad \text{in } \mathcal{D} \qquad w_{1,\varepsilon}=0, \qquad \text{on } \partial\mathcal{D}. \tag{3.39}$$

Moreover, it is straightforward to check that

$$\|h_{1,\varepsilon}\|_{\varepsilon,4,\sigma}\le C\varepsilon$$

for some $\sigma\in(0,1)$. Proposition 3.1 thus gives that

$$\|D^2_\xi w_{1,\varepsilon}\|_{\varepsilon,4,\sigma} + \|D_\xi w_{1,\varepsilon}\|_{\varepsilon,3,\sigma} + \|w_{1,\varepsilon}\|_{\varepsilon,2,\sigma}\le C\varepsilon \tag{3.40}$$

and that there exists a positive constant β (depending only on Ω, K and N) such that for any integer ℓ there holds

$$\|\nabla_z^{(\ell)}w_{1,\varepsilon}(z,\cdot)\|_{\varepsilon,2,\sigma}\le\beta C_l\varepsilon \qquad z\in K_\rho \tag{3.41}$$

where C_l depends only on l, p, K and Ω.

With this definition of $\mu_{0,\varepsilon}$, $d^0_{N,\varepsilon}$, $e_{0,\varepsilon}$ and $w_{1,\varepsilon}$, we have in particular that

$$\| -\mathcal{A}_{\mu_\varepsilon,d_\varepsilon}v_{1,\varepsilon} - \mu_\varepsilon^{\frac{N-2}{2}\varepsilon}v_{1,\varepsilon}^{p-\varepsilon}\|_{\varepsilon,4,\sigma}\le C\varepsilon^2.$$

Construction of $w_{2,\varepsilon}$ and choice of the parameters $\mu_{1,\varepsilon}$, $d^1_{N,\varepsilon}$ and $e_{1,\varepsilon}$. To improve further our approximate solutions $v_{1,\varepsilon}$ constructed in the previous step we define the function

$$v_{2,\varepsilon}(z,\xi)=U(\xi)-\bar U(\xi)+w_{1,\varepsilon}(z,\xi)+w_{2,\varepsilon}(z,\xi)+\varepsilon e_\varepsilon\chi_\varepsilon(\xi)Z_0,$$

where now $\mu_\varepsilon=\mu_{0,\varepsilon}+\mu_{1,\varepsilon}$, $d_{N,\varepsilon}=d^0_{N,\varepsilon}+d^1_{N,\varepsilon}$, $e_\varepsilon=e_{0,\varepsilon}+e_{1,\varepsilon}$ and where $\mu_{0,\varepsilon}$, $d^0_{N,\varepsilon}$, $e_{0,\varepsilon}$ and $w_{1,\varepsilon}$ have already been constructed in the previous step. Observe that a Taylor expansion yields

$$\begin{aligned}\bar U(\xi)=&U\left(\bar\xi,\xi_N+2\frac{\varepsilon(d^0_{N,\varepsilon}+d^1_{N,\varepsilon})}{\rho(\mu_{0,\varepsilon}+\mu_{1,\varepsilon})}\right)=U\big(\bar\xi,\xi_N+2\frac{\varepsilon d^0_{N,\varepsilon}}{\rho\mu_{0,\varepsilon}}\big)\\ &+2\frac{\varepsilon}{\rho}\partial_N U\big(\bar\xi,\xi_N+2\frac{\varepsilon d^0_{N,\varepsilon}}{\rho\mu_{0,\varepsilon}}\big)\left\{\frac{d^0_{N,\varepsilon}}{\mu_{0,\varepsilon}}\left(\frac{d^1_{N,\varepsilon}}{d^0_{N,\varepsilon}}-\frac{\mu_{1,\varepsilon}}{\mu_{0,\varepsilon}}\right)+O\left(\frac{d^1_{N,\varepsilon}}{d^0_{N,\varepsilon}}-\frac{\mu_{1,\varepsilon}}{\mu_{0,\varepsilon}}\right)^2\right\}.\end{aligned}\tag{3.42}$$

Computing $\mathcal S_\varepsilon(v_{2,\varepsilon})$ (see (2.18)) we get

$$\mathcal S_\varepsilon(v_{2,\varepsilon})=\mathcal L_\varepsilon w_{2,\varepsilon}+h_{2,\varepsilon}+\varepsilon^3\mathcal E_{2,\varepsilon}+Q_\varepsilon(w_{2,\varepsilon})\tag{3.43}$$

where $\mathcal L_\varepsilon$ is defined in (3.27) and the function $h_{2,\varepsilon}$ is given by

$$\begin{aligned}h_{2,\varepsilon}=&-2\varepsilon d^1_{N,\varepsilon}H_{ij}\partial^2_{ij}U+\rho\,\mu_{1,\varepsilon}\ \left[-2\xi_N H_{ij}\partial^2_{ij}U+H_{\alpha\alpha}\partial_N U\right]-\lambda_1\,\varepsilon\,e_{1,\varepsilon}\,Z_0\\ &-\varepsilon\frac{N-2}{2}\frac{\mu_{1,\varepsilon}}{\mu_{0,\varepsilon}}U^p+\tilde f_{2\varepsilon}+\tilde h_{2\varepsilon}(y,\xi,\mu_{0,\varepsilon},d^0_{N,\varepsilon},e_{0,\varepsilon})\end{aligned}\tag{3.44}$$

where

$$\tilde f_{2\varepsilon}=2pU^{p-1}\partial_N U\big(\bar\xi,\xi_N+2\frac{\varepsilon d^0_{N,\varepsilon}}{\rho\mu_{0,\varepsilon}}\big)\frac{\varepsilon d^0_{N,\varepsilon}}{\rho\mu_{0,\varepsilon}}\left[\frac{d^1_{N,\varepsilon}}{d^0_{N,\varepsilon}}-\frac{\mu_{1,\varepsilon}}{\mu_{0,\varepsilon}}\right],$$

and $\tilde h_{2\varepsilon}$ is a smooth function on its variables and is even in the variable $\bar\xi\in\mathbb R^{N-1}$, which implies in particular that

$$\int_{\hat{\mathcal D}}\tilde h_{2\varepsilon}Z_j d\xi=0\quad j=1,\dots,N-1,\tag{3.45}$$

Moreover we can easily show that

$$\int_{\hat{\mathcal D}}\tilde h_{2\varepsilon}Z_0 d\xi=\varepsilon^2\vartheta_1(\mu_{0,\varepsilon},d^0_{N,\varepsilon},e_{0,\varepsilon}),\quad \int_{\hat{\mathcal D}}\tilde h_{2\varepsilon}Z_{N+1}d\xi=\varepsilon^2\vartheta_2(\mu_{0,\varepsilon},d^0_{N,\varepsilon},e_{0,\varepsilon}),\tag{3.46}$$

and

$$\int_{\hat{\mathcal D}}\tilde h_{2\varepsilon}Z_N d\xi=\varepsilon\rho\vartheta_3(\mu_{0,\varepsilon},d^0_{N,\varepsilon},e_{0,\varepsilon}).\tag{3.47}$$

where $\vartheta_i(\mu_{0,\varepsilon},d^0_{N,\varepsilon},e_{0,\varepsilon}), i=1,2,3$ are some uniformly bounded functions. In (3.43) the term $\mathcal E_{2,\varepsilon}$ can be described as the sum of functions of the form (3.29). Finally the term $Q_\varepsilon(w_{2,\varepsilon})$ is a sum of terms depending on $w_{2,\varepsilon}$ like

$$\begin{aligned}&(\mu_{0,\varepsilon}+\mu_{1,\varepsilon})^{\frac{N-2}{2}\varepsilon}\left[-(U-\bar U+w_{1,\varepsilon}+w_{2,\varepsilon}+\varepsilon e_\varepsilon\chi_\varepsilon(\xi)Z_0)^{p-\varepsilon}\right.\\ &\left.+(U-\bar U+w_{1,\varepsilon}+\varepsilon e_\varepsilon\chi_\varepsilon(\xi)Z_0)^{p-\varepsilon}+(p-\varepsilon)(U-\bar U+w_{1,\varepsilon}+\varepsilon e_\varepsilon\chi_\varepsilon(\xi)Z_0)^{p-1-\varepsilon}w_{2,\varepsilon}\right]\end{aligned}$$

and linear terms in $w_{2,\varepsilon}$ multiplied by a term of order ε, like

$$(p-\varepsilon)\left((U-\bar U+w_{1,\varepsilon}+\varepsilon e_\varepsilon\chi_\varepsilon(\xi)Z_0)^{p-1-\varepsilon}-U^{p-1-\varepsilon}\right)w_{2,\varepsilon}.$$

First we define $\mu_{1,\varepsilon},d^1_{N,\varepsilon},e_{1,\varepsilon}$. Computations similar to those of (3.31)-(3.33) yield

$$\int_{\hat{\mathcal D}}h_{2\varepsilon}Z_{N+1}d\xi=-\varepsilon A_1\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-2}\left[(N-2)\left(\frac{\mu_{1,\varepsilon}}{\mu_{0,\varepsilon}}-\frac{d^1_{N,\varepsilon}}{d^0_{N,\varepsilon}}\right)+O\left(\varepsilon^{\frac{1}{N-2}}\right)\right](1+o(1))$$

$$\int_{\hat{\mathcal D}}h_{2\varepsilon}Z_N d\xi=\varepsilon^{1+\frac{1}{N-2}}A_3(N-1)\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1}\left(\frac{\mu_{1,\varepsilon}}{\mu_{0,\varepsilon}}-\frac{d^1_{N,\varepsilon}}{d^0_{N,\varepsilon}}\right)(1+o(1))$$

$$\int_{\hat{\mathcal{D}}} h_{2\varepsilon} Z_0 d\xi = \varepsilon A_4 \left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-2} \left[(N-2)\left(\frac{\mu_{1,\varepsilon}}{\mu_{0,\varepsilon}} - \frac{d^1_{N,\varepsilon}}{d^0_{N,\varepsilon}}\right) + O\left(\varepsilon^{\frac{1}{N-2}}\right)\right](1+o(1)).$$

We choose $\mu_{1,\varepsilon}, d^1_{N,\varepsilon}, e_{1,\varepsilon}$ so that

$$\int_{\hat{\mathcal{D}}} h_{2,\varepsilon} Z_l d\xi = 0, \quad l = 0, N, N+1. \tag{3.48}$$

We can easily see that the above orthogonality conditions are fulfilled provided we choose the parameters $\mu_{1,\varepsilon}, d^1_{N,\varepsilon}, e_{1,\varepsilon}$ to solve the following system

$$\begin{cases} (N-2)A_1 \frac{\mu_{0,\varepsilon}^{N-3}}{(d^0_{N,\varepsilon})^{N-2}} \mu_{1,\varepsilon} - (N-2)A_1 \frac{\mu_{0,\varepsilon}^{N-2}}{(d^0_{N,\varepsilon})^{N-1}} d_{N,\varepsilon} \\ +\varepsilon^{\frac{1}{N-2}} g_{N+1}\left(\frac{\mu_{1,\varepsilon}}{d^1_{N,\varepsilon}}\right) = \varepsilon \Re_1(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}) \\ \\ \left[A_6 H_{\alpha\alpha} - (N-1)A_3 \frac{\mu_{0,\varepsilon}^{N-2}}{(d^0_{N,\varepsilon})^{N-1}}\right] \mu_{1,\varepsilon} + (N-1)A_3 \frac{\mu_{0,\varepsilon}^{N-1}}{(d^0_{N,\varepsilon})^{N}} d^1_{N,\varepsilon} \\ +\varepsilon^{\frac{1}{N-2}} g_N \left(\frac{\mu_{1,\varepsilon}}{d^1_{N,\varepsilon}}\right) = \varepsilon \Re_2(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}) \\ \\ (N-2)A_4 \frac{\mu_{0,\varepsilon}^{N-2}}{(d^0_{N,\varepsilon})^{N-2}} \left(\frac{\mu_{1,\varepsilon}}{\mu_{0,\varepsilon}} - \frac{d^1_{N,\varepsilon}}{d^0_{N,\varepsilon}}\right) + A_5 - A_7 \log(\mu_{1,\varepsilon}) - \lambda_1 e_{1,\varepsilon} \\ -2H_{jj} d^1_{N,\varepsilon} \int_{\mathbb{R}^N} \partial^2_{jj} U Z_0 d\xi + \varepsilon^{\frac{1}{N-2}} g_0 \left(\frac{\mu_{1,\varepsilon}}{d^1_{N,\varepsilon}}\right) = \varepsilon \Re_3(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}) , \end{cases} \tag{3.49}$$

where $\Re_i$, $i = 1, 2, 3$ are some smooth uniformly bounded functions. Arguing as in the first step we can show that the above system is solvable and the solution $(\mu_{1,\varepsilon}, d^1_{N,\varepsilon}, e_{1,\varepsilon})$ has the form

$$\mu_{1,\varepsilon} = \tilde{\mu}_{1,\varepsilon} + \varepsilon^{\frac{1}{N-2}} \overline{\mu}_{1,\varepsilon}, \quad d^1_{N,\varepsilon} = \tilde{d}^1_{N,\varepsilon} + \varepsilon^{\frac{1}{N-2}} \overline{d}^1_{N,\varepsilon}, \quad e_{1,\varepsilon} = \tilde{e}_{1,\varepsilon} + \varepsilon^{\frac{1}{N-2}} \overline{e}_{1,\varepsilon}, \tag{3.50}$$

where $(\tilde{\mu}_{1,\varepsilon}, \tilde{d}^1_{N,\varepsilon}, \tilde{e}_{1,\varepsilon})$ is a solution of

$$\begin{cases} \tilde{\mu}_{1,\varepsilon} - \frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}} \tilde{d}^1_{N,\varepsilon} = \varepsilon \tilde{\Re}_1(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}) \\ \\ \left[A_6 H_{\alpha\alpha} - (N-1)A_3 \frac{\mu_{0,\varepsilon}^{N-2}}{(d^0_{N,\varepsilon})^{N-1}}\right] \tilde{\mu}_{1,\varepsilon} + (N-1)A_3 \frac{\mu_{0,\varepsilon}^{N-1}}{(d^0_{N,\varepsilon})^{N}} \tilde{d}^1_{N,\varepsilon} = \varepsilon \Re_2(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}) \\ \\ (N-2)A_4 \frac{\mu_{0,\varepsilon}^{N-2}}{(d^0_{N,\varepsilon})^{N-2}} \left(\frac{\tilde{\mu}_{1,\varepsilon}}{\mu_{0,\varepsilon}} - \frac{\tilde{d}^1_{N,\varepsilon}}{d^0_{N,\varepsilon}}\right) + A_5 - A_7 \log(\tilde{\mu}_{1,\varepsilon}) - \lambda_1 \tilde{e}_{1,\varepsilon} \\ -2H_{jj} \tilde{d}^1_{N,\varepsilon} \int_{\mathbb{R}^N} \partial^2_{jj} U Z_0 d\xi = \varepsilon \Re_3(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}) , \end{cases} \tag{3.51}$$

where $\tilde{\Re}_1 = \frac{1}{(N-2)A_1} \frac{(d^0_{N,\varepsilon})^{N-2}}{\mu_{0,\varepsilon}^{N-3}} \Re_1$. Indeed, the first two equations in (3.49) can be rewritten in the following form

$$M \begin{pmatrix} \tilde{\mu}_{1,\varepsilon} \\ \tilde{d}^1_{N,\varepsilon} \end{pmatrix} = \varepsilon \begin{pmatrix} \tilde{\Re}_1(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}) \\ \Re_2(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}) \end{pmatrix} \tag{3.52}$$

with the matrix

$$M = \begin{pmatrix} 1 & -\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}} \\ A_6 H_{\alpha\alpha} - (N-1)A_3 \frac{\mu_{0,\varepsilon}^{N-2}}{(d^0_{N,\varepsilon})^{N-1}} & (N-1)A_3 \frac{\mu_{0,\varepsilon}^{N-1}}{(d^0_{N,\varepsilon})^{N}} \end{pmatrix} \tag{3.53}$$

which is clearly invertible since $\det(M) = A_6 H_{\alpha\alpha} \frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}} \neq 0$. Thus we can get the existence of $\mu_{1,\varepsilon}$ and $d^1_{N,\varepsilon}$ in (3.51), and we then get the existence of $e_{1,\varepsilon}$ from the third equation in (3.51). Moreover, we have the following estimates

$$\|\mu_{1,\varepsilon}\|_{L^\infty(K)} + \|\partial_a \mu_{1,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a \mu_{1,\varepsilon}\|_{L^\infty(K)} \leq C\varepsilon,$$

$$\|d^1_{N,\varepsilon}\|_{L^\infty(K)} + \|\partial_a d^1_{N,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a d^1_{N,\varepsilon}\|_{L^\infty(K)} \leq C\varepsilon$$

and

$$\|e_{1,\varepsilon}\|_{L^\infty(K)} + \|\partial_a e_{1,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a e_{1,\varepsilon}\|_{L^\infty(K)} \leq C\varepsilon.$$

Observe now the following

- from (3.45) and using the fact that $\partial^2_{jj} U$ is even with respect to $\bar{\xi}$, we have

$$\int_{\hat{\mathcal{D}}} h_{2,\varepsilon} Z_j d\xi = 0, \quad j = 1, \ldots, N-1. \tag{3.54}$$

- given the choice of the parameters (3.50), the linear operator defined in (3.43) by (3.27), which depends on μ_ε, $d_{N,\varepsilon}$ and e_ε, satisfies the assumptions of Proposition 3.1.

Henceforth, we apply the result of Proposition 3.1 to define $w_{2,\varepsilon}$ solving

$$\mathcal{L}_\varepsilon w_{2,\varepsilon} = -h_{2,\varepsilon} \quad \text{in } \mathcal{D} \qquad w_{2,\varepsilon} = 0, \qquad \text{on } \partial\mathcal{D}. \tag{3.55}$$

Since, for a given $\sigma \in (0,1)$, $\|h_{2,\varepsilon}\|_{\varepsilon,4,\sigma} \leq C\varepsilon^2$, we have that

$$\|D^2_\xi w_{2,\varepsilon}\|_{\varepsilon,4,\sigma} + \|D_\xi w_{2,\varepsilon}\|_{\varepsilon,3,\sigma} + \|w_{2,\varepsilon}\|_{\varepsilon,2,\sigma} \leq C\varepsilon^2 \tag{3.56}$$

and that there exists a positive constant β (depending only on Ω, K and n) such that for any integer ℓ there holds

$$\|\nabla^{(\ell)}_y w_{2,\varepsilon}(z,\cdot)\|_{\varepsilon,2,\sigma} \leq \beta C_\ell \, \varepsilon^2 \qquad \rho y = z \in K_\rho \tag{3.57}$$

where C_ℓ depends only on ℓ, p, K and Ω.

With this choice of $\mu_{1,\varepsilon}$, $e_{1,\varepsilon}$, $d^1_{N,\varepsilon}$ and $w_{2,\varepsilon}$ we get that

$$\| - \mathcal{A}_{\mu_\varepsilon, d_\varepsilon} v_{2,\varepsilon} - \mu_\varepsilon^{\frac{N-2}{2}\varepsilon} v_{2,\varepsilon}^{p-\varepsilon} \|_{\varepsilon,4,\sigma} \leq C\varepsilon^3.$$

Construction of $w_{3,\varepsilon}$ and choice of $\mu_{2,\varepsilon}$, $d^2_{N,\varepsilon}$, $e_{2,\varepsilon}$ and $d^0_{j,\varepsilon}$, $l = 1, \ldots, N-1$.

We define

$$v_{3,\varepsilon}(z,\xi) = U(\xi) - \bar{U}\,(\xi) + w_{1,\varepsilon}\,(z,\xi) + w_{2,\varepsilon}\,(z,\xi) + w_{3,\varepsilon}\,(z,\xi) + \varepsilon e_\varepsilon \chi_\varepsilon(\xi) Z_0$$

where $\mu_\varepsilon = \mu_{0,\varepsilon} + \mu_{1,\varepsilon} + \mu_{2,\varepsilon}$, $e_\varepsilon = e_{0,\varepsilon} + e_{1,\varepsilon} + e_{2,\varepsilon}$, $d_{N,\varepsilon} = d^0_{N,\varepsilon} + d^1_{N,\varepsilon} + d^2_{N,\varepsilon}$, $d_{l,\varepsilon} = d^0_{1,\varepsilon}$, $l = 1, \ldots, N-1$. We recall that $\mu_{0,\varepsilon}, \mu_{1,\varepsilon}$, $e_{0,\varepsilon}, e_{1,\varepsilon}$, $d^0_{N,\varepsilon}, d^1_{N,\varepsilon}$ and $w_{1,\varepsilon}, w_{2,\varepsilon}$ have already been constructed in the previous steps. Computing $\mathcal{S}_\varepsilon(v_{3,\varepsilon})$ (see (2.18)) we get

$$\mathcal{S}_\varepsilon(v_{3,\varepsilon}) = \mathcal{L}_\varepsilon w_{3,\varepsilon} - h_{3,\varepsilon} + \varepsilon^4 \mathcal{E}_{3,\varepsilon} + Q_\varepsilon(w_{3,\varepsilon}) \tag{3.58}$$

where $\mathcal{L}_\varepsilon$ is defined in (3.27), and the function $h_{3,\varepsilon}$ is given by

$$\begin{aligned} h_{3,\varepsilon} = & -2\varepsilon d^2_{N,\varepsilon} H_{ij} \partial^2_{ij} U + \rho\, \mu_{2,\varepsilon} \, \left\{ -2\xi_N H_{ij} \partial^2_{ij} U + H_{\alpha\alpha} \partial_N U \right\} - \lambda_1 \, \varepsilon \, e_{2,\varepsilon} \, Z_0 - \varepsilon \frac{N-2}{2} \frac{\mu_{2,\varepsilon}}{\mu_{0,\varepsilon}} U^p \\ & + 2pU^{p-1} \partial_N U \big(\bar{\xi}, \xi_N + 2 \frac{\varepsilon d^0_{N,\varepsilon}}{\rho \mu_{0,\varepsilon}} \big) \frac{\varepsilon d^0_{N,\varepsilon}}{\rho \mu_{0,\varepsilon}} \left[\frac{d^2_{N,\varepsilon}}{d^0_{N,\varepsilon}} - \frac{\mu_{2,\varepsilon}}{\mu_{0,\varepsilon}} \right] + \varepsilon^2 \rho \Xi_3(d^0_{j,\varepsilon}) \\ & + \tilde{h}_{3\varepsilon}(y, \xi, \mu_{0,\varepsilon}, \mu_{1,\varepsilon}, d^0_{N,\varepsilon}, d^1_{N,\varepsilon}, e_{0,\varepsilon}, e_{1,\varepsilon}) \end{aligned} \tag{3.59}$$

with the function $\tilde{h}_{3,\varepsilon}$ satisfying

$$\int_{\hat{\mathcal{D}}} \tilde{h}_{3,\varepsilon} Z_j d\xi = O(\varepsilon^2 \rho), \quad j = 1, \ldots, N-1, \tag{3.60}$$

and

$$\int_{\hat{\mathcal{D}}} \tilde{h}_{3,\varepsilon} Z_{N+1} d\xi = O(\varepsilon^3), \qquad \int_{\hat{\mathcal{D}}} \tilde{h}_{3,\varepsilon} Z_N d\xi = O(\varepsilon^2 \rho), \qquad \int_{\hat{\mathcal{D}}} \tilde{h}_{3,\varepsilon} Z_0 d\xi = O(\varepsilon^3). \tag{3.61}$$

In (3.59), $\Xi_3(d^0_{j,\varepsilon})$ is given by

$$\begin{aligned}\Xi_3(d^0_{j,\varepsilon}) = \big\{ & -\mu_{0,\varepsilon}\partial_j U \triangle_K d^0_{j,\varepsilon} + \gamma(1+\gamma)\nabla_K \mu_{0,\varepsilon}\nabla_K d^0_{j,\varepsilon}\partial_j U + 2\nabla_K\mu_{0,\varepsilon}\nabla_K d^0_{j,\varepsilon}\partial^2_{jl}U\xi_l \\ & -2\mu_{0,\varepsilon}\tilde{g}^{ab}\frac{1}{\rho}\partial^2_{\bar{a}j}U\partial_b d^0_{j,\varepsilon} - \frac{1}{3}\mu_\varepsilon R_{mijl}(\xi_m d^0_{l,\varepsilon} + \xi_l d^0_{m,\varepsilon})\partial^2_{ij}U + \mu_{0,\varepsilon}\mathfrak{D}^{ij}_{Nl}\xi_N d^0_{l,\varepsilon}\partial^2_{ij}U \\ & +\mu_{0,\varepsilon}[\frac{2}{3}R_{mllj} + \tilde{g}^{ab}R_{jabm} - \Gamma^c_{am}\Gamma^a_{cj}]d^0_{m,\varepsilon}\partial_j v - 2\mu_{0,\varepsilon}\xi_N(H_{aj} + \tilde{g}^{ac}H_{cj})\partial_a d^0_{l,\varepsilon}\partial^2_{jl}U \Big\}.\end{aligned}$$

In (3.58) the term $\mathcal{E}_{3,\varepsilon}$ can be described as the sum of functions of the form (3.29).

Finally the term $Q_\varepsilon(w_{3,\varepsilon})$ is a sum of terms depending on $w_{2,\varepsilon}$ like

$$\begin{aligned}(\mu_{0,\varepsilon}+\mu_{1,\varepsilon}+\mu_{2,\varepsilon})^{\frac{N-2}{2}\varepsilon}\big[& (U-\bar{U}+w_{1,\varepsilon}+w_{2,\varepsilon}+w_{3,\varepsilon}+\varepsilon e_\varepsilon Z_0)^{p-\varepsilon} - (U-\bar{U}+w_{1,\varepsilon}+w_{2,\varepsilon}+\varepsilon e_\varepsilon Z_0)^{p-\varepsilon} \\ & -(p-\varepsilon)(U-\bar{U}+w_{1,\varepsilon}+w_{2,\varepsilon}+\varepsilon e_\varepsilon Z_0)^{p-1-\varepsilon}w_{3,\varepsilon}\big]\end{aligned}$$

and linear terms in $w_{3,\varepsilon}$ multiplied by a term of order ε, like

$$(p-\varepsilon)\left((U-\bar{U}+w_{1,\varepsilon}+w_{2,\varepsilon}+\varepsilon e_\varepsilon Z_0)^{p-1-\varepsilon} - U^{p-1-\varepsilon}\right)w_{3,\varepsilon}.$$

We now proceed with the choice of $\mu_{2,\varepsilon}, d^2_{N,\varepsilon}, e_{2,\varepsilon}$ and $d^0_{l,\varepsilon}$, $l=1,\dots,N-1$.

Projection onto Z_{N+1}, Z_N, Z_0 and choice of $\mu_{2,\varepsilon}, d^2_{N,\varepsilon}, e_{2,\varepsilon}$. Arguing as in the last step of the iteration we can prove that the three orthogonality conditions $\int_{\mathcal{D}} h_{3,\varepsilon}Z_l = 0, \quad l=0,N,N+1$. are guaranteed choosing the parameters $\mu_{2,\varepsilon}, d^2_{N,\varepsilon}, e_{2,\varepsilon}$, to be solutions of the following system

$$M\begin{pmatrix}\mu_{2,\varepsilon}\\ d^2_{N,\varepsilon}\end{pmatrix} = \varepsilon^2\begin{pmatrix}\tilde{\mathfrak{R}}_{13}(\mu_{0,\varepsilon},\mu_{1,\varepsilon};d^0_{N,\varepsilon},d^1_{N,\varepsilon};e_{0,\varepsilon},e_{1,\varepsilon})\\ \mathfrak{R}_{23}(\mu_{0,\varepsilon},\mu_{1,\varepsilon};d^0_{N,\varepsilon},d^1_{N,\varepsilon};e_{0,\varepsilon},e_{1,\varepsilon})\end{pmatrix}$$

and

$$\begin{aligned}&(N-2)A_4\frac{\mu^{N-2}_{0,\varepsilon}}{(d^0_{N,\varepsilon})^{N-2}}\left(\frac{\mu_{2,\varepsilon}}{\mu_{0,\varepsilon}} - \frac{d^2_{N,\varepsilon}}{d^0_{N,\varepsilon}}\right) + A_5 - A_7\log(\mu_{2,\varepsilon}) - \lambda_1 e_{2,\varepsilon} \\ - \quad & 2H_{jj}d^2_{N,\varepsilon}\int_{\mathbb{R}^N}\partial^2_{jj}UZ_0 d\xi = \varepsilon^2\mathfrak{R}_{33}(\mu_{0,\varepsilon},\mu_{1,\varepsilon};d^0_{N,\varepsilon},d^1_{N,\varepsilon};e_{0,\varepsilon},e_{1,\varepsilon}).\end{aligned}$$

where the matrix M was defined in (3.53). Arguing as in the previous step, we can get the existence and smoothness of $\mu_{2,\varepsilon}, d^2_{N,\varepsilon}$, $e_{2,\varepsilon}$, solutions of the above system. Moreover, we have the validity of the following bounds on such parameters

$$\|\mu_{2,\varepsilon}\|_{L^\infty(K)} + \|\partial_a\mu_{2,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a\mu_{2,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^2,$$
$$\|d^2_{N,\varepsilon}\|_{L^\infty(K)} + \|\partial_a d^2_{N,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a d^2_{N,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^2,$$

and

$$\|e_{2,\varepsilon}\|_{L^\infty(K)} + \|\partial_a e_{2,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a e_{2,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^2.$$

Projection onto Z_l and choice of $d^0_{l,\varepsilon}$: Multiplying $h_{3,\varepsilon}$ by $Z_l = \partial_l U$, integrating over $\mathcal{D}$ and using the fact U is even in the variable $\bar{\xi}$, one obtains

$$\begin{aligned}\int_{\hat{\mathcal{D}}} h_{3,\varepsilon}\, Z_l = \varepsilon^2\rho\Big\{ & -\mu_{0,\varepsilon}\,\Delta_K d^0_{j,\varepsilon}\int_{\hat{\mathcal{D}}}\partial_j U\partial_l U + \varepsilon\int_{\hat{\mathcal{D}}}\mathfrak{G}_{2,\varepsilon}\,\partial_l U \\ & -\frac{1}{3}\,\mu_{0,\varepsilon}\,R_{mijs}\int_{\hat{\mathcal{D}}}(\xi_m d^0_{s,\varepsilon}+\xi_s d^0_{m,\varepsilon})\,\partial^2_{ij}U\partial_l U \\ & +\mu_{0,\varepsilon}\left[\frac{2}{3}\,R_{mssj}\,d^0_{m,\varepsilon} + \left(\tilde{g}^{ab}_\varepsilon\,R_{maaj} - \Gamma^c_a(E_m)\Gamma^a_c(E_j)\right)d^0_{m,\varepsilon}\right]\int_{\hat{\mathcal{D}}}\partial_j U\,\partial_l U\Big\} + O(\varepsilon^2\rho).\end{aligned} \tag{3.62}$$

First of all, observe that by oddness in $\bar{\xi}$ we have that

$$\int_{\hat{\mathcal{D}}} \partial_j U \partial_l U = \delta_{lj} \left(\int_{\mathbb{R}^N} |\partial_l U|^2 + O(\varepsilon^{N-2}) \right) = \delta_{lj}\, C_0 + O(\varepsilon^{N-2})$$

with $C_0 := \int_{\mathbb{R}^N} |\partial_l w_0|^2$. On the other hand the integral $\int_{\hat{\mathcal{D}}} \xi_m\, \partial^2_{ij} U \partial_l U$ is non-zero only if, either $i = j$ and $m = l$, or $i = l$ and $j = m$, or $i = m$ and $j = l$. In the latter case we have $R_{mijs} = 0$ (by the antisymmetry of the curvature tensor in the first two indices). Therefore, the first term of the second line of the above formula becomes simply

$$\begin{aligned}
&\sum_{ijms} R_{mijs} \int_{\hat{\mathcal{D}}} \xi_m d^0_{s,\varepsilon}\, \partial^2_{ij} U \partial_l U \\
= &\sum_{is} R_{liis} d^0_{s,\varepsilon} \int_{\hat{\mathcal{D}}} \xi_l \partial_l U \partial^2_{ii} U d\xi + \sum_{js} R_{jljs} d^0_{s,\varepsilon} \int_{\hat{\mathcal{D}}} \xi_j \partial_l U \partial^2_{lj} U d\xi \\
= &\sum_{is} R_{liis} d^0_{s,\varepsilon} \int_{\mathbb{R}^N} \xi_l \partial_l U \partial^2_{ii} U d\xi + \sum_{js} R_{jljs} d^0_{s,\varepsilon} \int_{\mathbb{R}^N} \xi_j \partial_l U \partial^2_{lj} U d\xi + O(\varepsilon^{N-2}).
\end{aligned}$$

Observe that, integrating by parts, when $l \neq i$ (otherwise $R_{liis} = 0$) there holds

$$\int_{\hat{\mathcal{D}}} \xi_l \partial_l U \partial^2_{ii} U d\xi = \int_{\mathbb{R}^N} \xi_l \partial_l U \partial^2_{ii} U d\xi + O(\varepsilon^{N-2}) = -\int_{\mathbb{R}^N} \xi_l \partial_i U \partial^2_{li} U d\xi + O(\varepsilon^{N-2}).$$

Hence, still by the antisymmetry of the curvature tensor we obtain that

$$\sum_{ijms} R_{mijs} \int_{\hat{\mathcal{D}}} \xi_m d^0_{s,\varepsilon}\, \partial^2_{ij} U \partial_l U = -2 \sum_{is} R_{liis} d^0_{s,\varepsilon} \Big(\int_{\mathbb{R}^N} \xi_l \partial_i U \partial^2_{li} U d\xi + O(\varepsilon^{N-2}) \Big).$$

Then the second line in Formula (3.62) becomes (permuting the indices s and m in the above argument)

$$\begin{aligned}
&-\frac{1}{3}\, \mu_{0,\varepsilon} \sum_{ijms} R_{mijs} \int_{\hat{\mathcal{D}}} (\xi_m d^0_{s,\varepsilon} + \xi_s d^0_{m,\varepsilon})\, \partial^2_{ij} U \partial_l U \\
= \;&\frac{4}{3}\, \mu_{0,\varepsilon} \sum_{is} R_{liis} d^0_{s,\varepsilon} \Big(\int_{\mathbb{R}^N} \xi_l \partial_i U \partial^2_{li} U d\xi + O(\varepsilon^{N-2}) \Big) = -\frac{2}{3}\, \mu_{0,\varepsilon} \sum_{is} R_{liis} d^0_{s,\varepsilon} \Big(C_0 + O(\varepsilon^{N-2}) \Big)
\end{aligned}$$

Collecting the above computations, we conclude that

$$-\frac{1}{3}\, \mu_{0,\varepsilon}\, R_{mijl} \int_{\hat{\mathcal{D}}} (\xi_m d^0_{l,\varepsilon} + \xi_l d^0_{m,\varepsilon})\, \partial^2_{ij} U \partial_l U + \frac{2}{3}\, \mu_{0,\varepsilon}\, R_{mssj}\, d^0_{m,\varepsilon} \int_{\hat{\mathcal{D}}} \partial_j U\, \partial_l U = O(\varepsilon^{N-2}).$$

Hence formula (3.62) becomes simply

$$\begin{aligned}
[\mu_{0,\varepsilon} \varepsilon^2 \rho]^{-1} \int_{\hat{\mathcal{D}}} h_{3,\varepsilon}\, \partial_l U = &- C_0\, \Delta_K\, d^0_{l,\varepsilon} + C_0 \left(\tilde{g}^{ab}_{\varepsilon}\, R_{maal} - \Gamma^c_a(E_m) \Gamma^a_c(E_l) + O(\varepsilon^{N-2}) \right) d^0_{m,\varepsilon} \\
&+ \int_{\hat{\mathcal{D}}} \mathfrak{G}_{2,\varepsilon}\, \partial_l w_0.
\end{aligned}$$

We thus obtain that $h_{3,\varepsilon}$, the right-hand side of (3.55), is L^2-orthogonal to Z_l ($l = 1, \cdots, N-1$) if and only if $d^0_{l,\varepsilon}$ satisfies an equation of the form

$$\Delta_K\, d^0_{l,\varepsilon} - \left(\tilde{g}^{ab}_{\varepsilon}\, R_{maal} - \Gamma^c_a(E_m) \Gamma^a_c(E_l) + O(\varepsilon^{N-2}) \right) d^0_{m,\varepsilon} = G_{2,\varepsilon}(\rho z), \tag{3.63}$$

for some smooth function $G_{2,\varepsilon}$, whose L^∞ norm on K is bounded by a fixed constant, as $\varepsilon \to 0$. Observe that the operator acting on $d^0_{l,\varepsilon}$ in the left hand side is nothing but the Jacobi operator of the submanifold K. By our assumption, K is non degenerate and hence this operator is invertible. This implies the solvability of the above equation in $d^0_{l,\varepsilon}$. Furthermore, equation (3.63) defines $d^0_{l,\varepsilon}$ as a smooth function on K, with

$$\|d^0_{l,\varepsilon}\|_{L^\infty(K)} + \|\partial_a d^0_{l,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a d^0_{l,\varepsilon}\|_{L^\infty(K)} \le C \quad l = 1, \ldots, N-1. \tag{3.64}$$

Given the choice of the parameters $\mu_{2,\varepsilon}, d^2_{N,\varepsilon}$, $e_{2,\varepsilon}$ and $d^0_{l,\varepsilon}(l=1,\dots,N-1)$, the linear operator defined in (3.58) by (3.27), which depends on μ_ε, $d_{N,\varepsilon}$ and e_ε, satisfies the assumptions of Proposition 3.1. Furthermore, we have the existence of $w_{3,\varepsilon}$ solution to

$$\mathcal{L}_\varepsilon w_{3,\varepsilon} = h_{3,\varepsilon} \quad \text{in } \mathcal{D}, \qquad w_{3,\varepsilon}=0, \quad \text{on } \partial\mathcal{D}. \tag{3.65}$$

Moreover, for a given $\sigma\in(0,1)$ we have $\|h_{3,\varepsilon}\|_{\varepsilon,4,\sigma}\le C\varepsilon^3$. Proposition 3.1 thus gives then that

$$\|D^2_\xi w_{3,\varepsilon}\|_{\varepsilon,4,\sigma}+\|D_\xi w_{3,\varepsilon}\|_{\varepsilon,3,\sigma}+\|w_{3,\varepsilon}\|_{\varepsilon,2,\sigma}\le C\varepsilon^3 \tag{3.66}$$

and that there exists a positive constant β (depending only on Ω, K and n) such that for any integer ℓ there holds

$$\|\nabla^{(\ell)}_y w_{3,\varepsilon}(z,\cdot)\|_{\varepsilon,N-3,\sigma}\le \beta C_\ell\,\varepsilon^3 \qquad y=\rho z\in K. \tag{3.67}$$

where C_ℓ depends only on ℓ, p, K and Ω. Moroever, we have that

$$\|-\mathcal{A}_{\mu_\varepsilon,d_\varepsilon}v_{3,\varepsilon}-\mu_\varepsilon^{\frac{N-2}{2}\varepsilon}\,v_{3,\varepsilon}^{p-\varepsilon}\|_{\varepsilon,4,\sigma}\le C\varepsilon^4.$$

Expansion at an arbitrary order. We take now an arbitrary integer I, we let

$$\mu_\varepsilon := \mu_{0,\varepsilon}+\mu_{1,\varepsilon}+\cdots+\mu_{I-1,\varepsilon}+\mu_{I,\varepsilon}, \tag{3.68}$$

$$d_{l,\varepsilon}=d^0_{l,\varepsilon}+\ldots+d^{I-2}_{l,\varepsilon}. \quad l=1,\dots,N-1; \qquad d_{N,\varepsilon}=d^0_{N,\varepsilon}+\ldots+d^I_{N,\varepsilon}$$

and

$$e_\varepsilon = e_{0,\varepsilon}+e_{1,\varepsilon}+\cdots+e_{I,\varepsilon}$$

and we define

$$v_{I+1,\varepsilon}=U(\xi)-\bar U(\xi)+w_{1,\varepsilon}(z,\xi)+\ldots+w_{I,\varepsilon}(z,\xi)+w_{I+1,\varepsilon}(z,\xi)+\varepsilon e_\varepsilon\chi_\varepsilon Z_0 \tag{3.69}$$

where $\mu_{0,\varepsilon},\mu_{1,\varepsilon},\cdots,\mu_{I-1,\varepsilon}$, $d^1_{l,\varepsilon},\cdots,d^{I-3}_{l,\varepsilon}$, $d^0_{N,\varepsilon},\dots,d^{I-1}_{N,\varepsilon}$, $e_{0,\varepsilon},e_{1,\varepsilon},\dots,e_{I-1,\varepsilon}$ and $w_{1,\varepsilon},..,w_{I,\varepsilon}$ have already been constructed following an iterative scheme, as described in the previous steps of the construction.

In particular one has, for any $i=1,\dots,I-1$,

$$\|\mu_{i,\varepsilon}\|_{L^\infty(K)}+\|\partial_a\mu_{i,\varepsilon}\|_{L^\infty(K)}+\|\partial^2_a\mu_{i,\varepsilon}\|_{L^\infty(K)}\le C\varepsilon^i,$$

$$\|d^i_{N,\varepsilon}\|_{L^\infty(K)}+\|\partial_a d^i_{N,\varepsilon}\|_{L^\infty(K)}+\|\partial^2_a d^i_{N,\varepsilon}\|_{L^\infty(K)}\le C\varepsilon^{i-1},$$

$$\|d^i_{l,\varepsilon}\|_{L^\infty(K)}+\|\partial_a d^i_{l,\varepsilon}\|_{L^\infty(K)}+\|\partial^2_a d^i_{l,\varepsilon}\|_{L^\infty(K)}\le C\varepsilon^{i-1}, \quad l=1,\dots,N-1,$$

$$\|e_{i,\varepsilon}\|_{L^\infty(K)}+\|\partial_a e_{i,\varepsilon}\|_{L^\infty(K)}+\|\partial^2_a e_{i,\varepsilon}\|_{L^\infty(K)}\le C\varepsilon^{i-1},$$

and moreover for any $i=0,\dots,I-1$ we have that

$$\|D^2_\xi w_{i+1,\varepsilon}\|_{\varepsilon,4,\sigma}+\|D_\xi w_{i+1,\varepsilon}\|_{\varepsilon,3,\sigma}+\|w_{i+1,\varepsilon}\|_{\varepsilon,2,\sigma}\le C\varepsilon^{i+1}$$

and for any integer ℓ

$$\|\nabla^{(\ell)}_z w_{i+1,\varepsilon}(z,\cdot)\|_{\varepsilon,2,\sigma}\le\beta C_l\varepsilon^{i+1}, \qquad z\in K_\rho.$$

The new components $(\mu_{I,\varepsilon}, d^{I-2}_{1,\varepsilon},\dots,d^{I-2}_{N-1,\varepsilon},d^I_{N,\varepsilon},e_{I,\varepsilon})$ will be found reasoning as before. Computing $S(v_{I+1,\varepsilon})$ (see (2.18)) we get

$$\mathcal{S}_\varepsilon(v_{I+1,\varepsilon})=\mathcal{L}_\varepsilon w_{I+1,\varepsilon}-h_{I+1,\varepsilon}+\varepsilon^{I+2}\mathcal{E}_{I+1,\varepsilon}+Q_\varepsilon(w_{I+1,\varepsilon}) \tag{3.70}$$

where $\mathcal{L}_\varepsilon$ is defined in (3.27), and the function $h_{3,\varepsilon}$ is given by

$$\begin{aligned} h_{I+1,\varepsilon} \;=\;& -2\varepsilon d^I_{N,\varepsilon}H_{ij}\partial^2_{ij}U+\rho\,\mu_{I,\varepsilon}\,\left\{-2\xi_N H_{ij}\partial^2_{ij}U+H_{\alpha\alpha}\partial_N U\right\}-\lambda_1\,\varepsilon\,e_{I,\varepsilon}\,Z_0\\ &-\varepsilon\frac{N-2}{2}\frac{\mu_{I,\varepsilon}}{\mu_{0,\varepsilon}}U^p+2pU^{p-1}\partial_N U\big(\bar\xi,\xi_N+2\frac{\varepsilon d^0_{N,\varepsilon}}{\rho\mu_{0,\varepsilon}}\big)\,\frac{\varepsilon d^0_{N,\varepsilon}}{\rho\mu_{0,\varepsilon}}\left[\frac{d^I_{N,\varepsilon}}{d^0_{N,\varepsilon}}-\frac{\mu_{I,\varepsilon}}{\mu_{0,\varepsilon}}\right]\\ &+\varepsilon^I\,\rho\,\Xi_{I+1}(d^{I-2}_{j,\varepsilon})+\tilde h_{I+1,\varepsilon}\end{aligned} \tag{3.71}$$

where $\tilde h_{I+1,\varepsilon}$ is a smooth function on its variable which depends only on the parameters $\mu_{j,\varepsilon}$, $d^j_{N,\varepsilon}$, $d^j_{\ell,\varepsilon}$, $e_{j,\varepsilon}$ which have been constructed in the previous steps. with

$$\int_{\hat{\mathcal{D}}} \tilde h_{I+1,\varepsilon} Z_j d\xi = O(\varepsilon^I \rho), \quad j = 1, \ldots, N-1, \tag{3.72}$$

and

$$\int_{\hat{\mathcal{D}}} \tilde h_{I+1,\varepsilon} Z_{N+1} d\xi = O(\varepsilon^{I+1}), \qquad \int_{\hat{\mathcal{D}}} \tilde h_{I+1,\varepsilon} Z_N d\xi = O(\varepsilon^I \rho), \qquad \int_{\hat{\mathcal{D}}} \tilde h_{I+1,\varepsilon} Z_0 d\xi = O(\varepsilon^I \rho). \tag{3.73}$$

In (3.59), $\Xi_{I+1}(d^{I-2}_{j,\varepsilon})$ is given by

$$\begin{aligned}\Xi_{I+1}(d^{I-2}_{j,\varepsilon}) =& -\mu_{0,\varepsilon}\partial_j U \triangle_K d^{I-2}_{j,\varepsilon} + \gamma(1+\gamma)\nabla_K \mu_{0,\varepsilon}\nabla_K d^{I-2}_{j,\varepsilon}\partial_j U + 2\nabla_K\mu_{0,\varepsilon}\nabla_K d^{I-2}_{j,\varepsilon}\partial^2_{jl}U\xi_l \\ & - 2\mu_{0,\varepsilon}\tilde g^{ab}\frac{1}{\rho}\partial^2_{\bar a j}U\partial_b d^j_{I-1,\varepsilon} - \frac{2}{3}\mu_\varepsilon R_{islj}\xi_s d^{I-2}_{l,\varepsilon}\partial^2_{ij}U + \mu_{0,\varepsilon}\mathfrak{D}^{ij}_{Nl}\xi_N d^{I-2}_{l,\varepsilon}\partial^2_{ij}U \\ & + \mu_{0,\varepsilon}[\frac{2}{3}R_{mllj} + \tilde g^{ab}R_{jabm} - \Gamma^c_{am}\Gamma^a_{cj}]d^{I-2}_{m,\varepsilon}\partial_j v - 2\mu_{0,\varepsilon}\xi_N(H_{aj} + \tilde g^{ac}H_{cj})\partial_a d^{I-2}_{l,\varepsilon}\partial^2_{jl}U.\end{aligned}$$

In (3.70) the term $\mathcal{E}_{I+1,\varepsilon}$ can be described as the sum of functions of the form (3.29). Finally the term $Q_\varepsilon(w_{I+1,\varepsilon})$ in (3.71) is a sum of terms depending on $w_{I+1,\varepsilon}$ like

$$(\mu_{0,\varepsilon} + \mu_{1,\varepsilon} + \cdots + \mu_{I-1,\varepsilon} + \mu_{I,\varepsilon})^{\frac{N-2}{2}\varepsilon}\left[v^{p-\varepsilon}_{I+1,\varepsilon} - v^{p-\varepsilon}_{I,\varepsilon} - (p-\varepsilon)v^{p-1-\varepsilon}_{I,\varepsilon}w_{I+1,\varepsilon}\right]$$

and linear terms in $w_{I+1,\varepsilon}$ multiplied by a term of order ε^2, like

$$(p-\varepsilon)\left((U-\bar U + w_{1,\varepsilon})^{p-1-\varepsilon} - (U-\bar U)^{p-1-\varepsilon}\right)w_{I+1,\varepsilon}.$$

Arguing as in the previous step, it is possible to prove the existence of parameters $\mu_{I,\varepsilon}$ and the normal section $d^{I-2}_{1,\varepsilon}, \ldots, d^{I-2}_{N-1,\varepsilon}, d^I_{N,\varepsilon}$ and $e_{I,\varepsilon}$ in such a way that $h_{I+1,\varepsilon}$ is L^2-orthogonal to Z_j, $j = 0, 1, \cdots, N+1$. Furthermore,

$$\|\mu_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial_a\mu_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a\mu_{I,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^I,$$

$$\|d^I_{N,\varepsilon}\|_{L^\infty(K)} + \|\partial_a d^I_{N,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a d^I_{N,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^I,$$

$$\|e_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial_a e_{I,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a e_{I,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^I.$$

and

$$\|d^{I-2}_{l,\varepsilon}\|_{L^\infty(K)} + \|\partial_a d^{I-2}_{l,\varepsilon}\|_{L^\infty(K)} + \|\partial^2_a d^{I-2}_{l,\varepsilon}\|_{L^\infty(K)} \le C\varepsilon^{I-2}. \tag{3.74}$$

We are now in a position to apply Proposition 3.1 to get a solution $w_{I+1,\varepsilon}$ to

$$\mathcal{L}_\varepsilon w_{I+1,\varepsilon} = h_{I+1,\varepsilon} \quad \text{in } \mathcal{D} \qquad w_{I+1,\varepsilon} = 0, \quad \text{on } \partial\mathcal{D}. \tag{3.75}$$

where $\mathcal{L}_\varepsilon$ is defined in (3.27). Furthermore, we have that

$$\|D^2_\xi w_{I+1,\varepsilon}\|_{\varepsilon,4,\sigma} + \|D_\xi w_{I+1,\varepsilon}\|_{\varepsilon,3,\sigma} + \|w_{I+1,\varepsilon}\|_{\varepsilon,2,\sigma} \le C\varepsilon^{I+1} \tag{3.76}$$

and that there exists a positive constant β (depending only on Ω, K and N) such that for any integer ℓ there holds

$$\|\nabla^{(\ell)}_y w_{I+1,\varepsilon}(z,\cdot)\|_{\varepsilon,2,\sigma} \le \beta C_l \varepsilon^{I+1} \qquad y = \rho z \in K. \tag{3.77}$$

With this choice of $(\mu_{I,\varepsilon}, d^{I-2}_{1,\varepsilon}, \ldots, d^{I-2}_{N-1,\varepsilon}, d^I_{N,\varepsilon}, e_{I,\varepsilon})$ and $w_{I+1,\varepsilon}$ we obtain that

$$\| - \mathcal{A}_{\mu_\varepsilon, d_\varepsilon} v_{I+1,\varepsilon} - \mu_\varepsilon^{\frac{N-2}{2}\varepsilon} v^{p-\varepsilon}_{I+1,\varepsilon}\|_{\varepsilon,4,\sigma} \le C\varepsilon^{I+2}.$$

This concludes our construction and have the validity of Proposition 3.2.

4. The existence result: Proof of the Theorem 1

Let us recall that if u is a solution to problem (1.3), and defining $\tilde{u}$ by

$$u(x) = (1+\alpha_\varepsilon)\rho^{-\frac{N-2}{2}}\tilde{u}(\rho^{-1}x),$$

then $\tilde{u}$ satisfies the following equation

$$-\Delta\tilde{u} = \tilde{u}^{\frac{N+2}{N-2}-\varepsilon}, \quad \tilde{u}>0 \quad \text{in } \Omega_\rho; \qquad \tilde{u}=0 \quad \text{on} \quad \partial\Omega_\rho, \tag{4.1}$$

where $\Omega_\rho = \frac{\Omega}{\rho}$.

4.1. **Global approximate solution.** Let I be an integer and recall the definitions of $K_\rho = \frac{1}{\rho}K$ and $\hat{\mathcal{D}}$ given after (2.13). We have constructed an approximate solution $v_{I+1,\varepsilon}$ in Section 3, such that

$$\mathcal{S}_\varepsilon(v_{I+1,\varepsilon}) = -\mathcal{A}_{\mu_\varepsilon,d_\varepsilon}v_{I+1,\varepsilon} - \mu_\varepsilon^{\frac{N-2}{2}\varepsilon}v_{I+1,\varepsilon}^{\frac{N+2}{N-2}-\varepsilon} = \mathcal{O}(\varepsilon^{I+2}) \quad \text{in} \quad K_\rho\times\hat{\mathcal{D}},$$

where $\mu_\varepsilon(y)$ and $d_\varepsilon(y)$ are functions defined on K, whose existence and properties are established in Proposition 3.2. We define locally around K_ρ the function

$$\begin{aligned}\tilde{U}_\varepsilon(z,x) \quad := \quad & \mu_\varepsilon^{-\frac{N-2}{2}}(\rho z)\, v_{I+1,\varepsilon}\left(z,\, \frac{\bar{x}-\varepsilon^2\rho^{-1}\bar{d}_\varepsilon(\rho z)}{\mu_\varepsilon(\rho z)}\,, \frac{x_N-\varepsilon\rho^{-1}d_{N,\varepsilon}(\rho z)}{\mu_\varepsilon(\rho z)}\right)\\ & \times\chi_\varepsilon(|(\bar{x}-\varepsilon^2\rho^{-1}\bar{d}_\varepsilon, x_N-\varepsilon\rho^{-1}d_{N,\varepsilon})|)\end{aligned} \tag{4.2}$$

where $z\in K_\rho$. Here χ_ε is a smooth cut-off function with

$$\chi_\varepsilon(r)=1, \text{ for } r\in[0,2\varepsilon^{-\gamma}], \quad \chi_\varepsilon(r)=0, \text{ for } r\in[3\varepsilon^{-\gamma},4\varepsilon^{-\gamma}], \quad \text{and} \quad |\chi_\varepsilon^{(l)}(r)|\le C_l\varepsilon^{l\gamma}, \quad \forall l\ge 1, \tag{4.3}$$

for some $\gamma\in(\frac{1}{2},1)$ to be fixed later.

We will use the notation

$$\tilde{u} = \tilde{\mathcal{T}}_{\mu_\varepsilon,d_\varepsilon}(\tilde{v}) \tag{4.4}$$

if and only if $\tilde{u}$ and $\tilde{v}$ satisfy

$$\tilde{u} = \mu_\varepsilon^{-\frac{N-2}{2}}(\rho z)\,\tilde{v}\left(z,\, \frac{\bar{x}-\varepsilon^2\rho^{-1}\bar{d}_\varepsilon(\rho z)}{\mu_\varepsilon(\rho z)}\,, \frac{x_N-\varepsilon\rho^{-1}d_{N,\varepsilon}(\rho z)}{\mu_\varepsilon(\rho z)}\right).$$

The function $\tilde{U}_\varepsilon$ is globally defined in Ω_ρ. We will look for a solution to (4.1) of the form

$$\tilde{u}_\varepsilon = \tilde{U}_\varepsilon + \phi,$$

where ϕ is a lower term. Thus ϕ satisfies the following problem

$$L_\varepsilon(\phi) := -\Delta\phi - (p-\varepsilon)\tilde{U}_\varepsilon^{p-1-\varepsilon}\phi = S_\varepsilon(\tilde{U}_\varepsilon) + N_\varepsilon(\phi) \quad \text{in } \Omega_\rho, \quad \phi=0 \quad \text{on} \quad \partial\Omega_\rho, \tag{4.5}$$

where

$$S_\varepsilon(\tilde{U}_\varepsilon) = \Delta_{g^\rho}\tilde{U}_\varepsilon + \tilde{U}_\varepsilon^p, \tag{4.6}$$

and

$$N_\varepsilon(\phi) = (\tilde{U}_\varepsilon+\phi)^{p-\varepsilon} - \tilde{U}_\varepsilon^p - (p-\varepsilon)\tilde{U}_\varepsilon^{p-1-\varepsilon}\phi, \tag{4.7}$$

where $g^\rho(y,x) = g(\rho y, \rho x)$.

To solve the Non-Linear Problem (4.5) we use a fixed point argument based on the contraction Mapping Principle. First we establish some invertibility properties of the linear problem

$$L_\varepsilon(\phi) = f \quad \text{in} \quad \Omega_\rho, \quad \phi=0 \quad \text{on} \quad \partial\Omega_\rho$$

with $f\in L^2(\Omega_\rho)$. This is the purpose of the next result.

Proposition 4.1. *There exist a sequence $\varepsilon_l\to 0$ and a positive constant $C>0$, such that, for any $f\in L^2(\Omega_{\rho_l})$, there exists a solution $\phi\in H_0^1(\Omega_{\rho_l})$ to the equation*

$$L_{\varepsilon_l}\phi = f \quad \text{in } \Omega_{\rho_l}, \quad \phi=0 \quad \text{on } \partial\Omega_{\rho_l},$$

with $\rho_l = \varepsilon_l^{\frac{N-1}{N-2}}$. Furthermore,

$$\|\phi\|_{H_0^1(\Omega_{\rho_l})} \le C\,\rho_l^{-\max\{2,k\}}\|f\|_{L^2(\Omega_{\rho_l})}. \tag{4.8}$$

The proof of this proposition will be given in Section 5. We are now in position to prove our main Theorem 1.

4.2. **Proof of the main Theorem 1.** By Proposition 4.1, $\phi \in H_0^1(\Omega_\rho)$ is a solution to (4.5) if and only if

$$\phi = L_\varepsilon^{-1}\left(S_\varepsilon(\tilde{U}_\varepsilon) + N_\varepsilon(\phi)\right).$$

Notice that

$$\|N_\varepsilon(\phi)\|_{L^2(\Omega_\rho)} \leq C \begin{cases} \|\phi\|^p_{H_0^1(\Omega_\rho)} & \text{for } p \leq 2, \\ \|\phi\|^2_{H_0^1(\Omega_\rho)} & \text{for } p > 2 \end{cases} \qquad \|\phi\|_{H_0^1(\Omega_\rho)} \leq 1 \tag{4.9}$$

and

$$\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{L^2(\Omega_\rho)} \leq C \begin{cases} \left(\|\phi_1\|^{p-1}_{H_0^1(\Omega_\rho)} + \|\phi_2\|^{p-1}_{H^1_{g^\varepsilon}(\Omega_\rho)}\right)\|\phi_1 - \phi_2\|_{H_0^1(\Omega_\rho)} & \text{for } p \leq 2, \\ \left(\|\phi_1\|_{H_0^1(\Omega_\rho)} + \|\phi_2\|_{H_0^1(\Omega_\rho)}\right)\|\phi_1 - \phi_2\|_{H_0^1(\Omega_\rho)} & \text{for } p > 2 \end{cases} \tag{4.10}$$

for any ϕ_1, ϕ_2 in $H_0^1(\Omega_\rho)$ with $\|\phi_1\|_{H_0^1(\Omega_\rho)}$, $\|\phi_2\|_{H_0^1(\Omega_\rho)} \leq 1$.

Defining $T_\varepsilon : H_0^1(\Omega_\rho) \to H_0^1(\Omega_\rho)$ as

$$T_\varepsilon(\phi) = L_\varepsilon^{-1}\left(S_\varepsilon(\tilde{U}_\varepsilon) + N_\varepsilon(\phi)\right)$$

we will show that T_ε is a contraction in some small ball in $H_0^1(\Omega_\rho)$. As a direct consequence of (3.26), we have $\|S_\varepsilon(\tilde{U}_\varepsilon)\|_{L^2(\Omega_\rho)} \leq C\varepsilon^{I+1}$. Using this inequality and by (4.9), (4.10) and (4.8), we obtain

$$\|T_\varepsilon(\phi)\|_{H^1(\Omega_\rho)} \leq C\rho^{-\max\{2,k\}} \begin{cases} \left(\varepsilon^{I+1} + \|\phi\|^p_{H_0^1(\Omega_\rho)}\right) & \text{for } p \leq 2, \\ \left(\varepsilon^{I+1} + \|\phi\|^2_{H_0^1(\Omega_\rho)}\right) & \text{for } p > 2. \end{cases}$$

Now we choose integers d and I so that

$$d > \begin{cases} \frac{N-1}{N-2}\frac{\max\{2,k\}}{p-1} & \text{for } p \leq 2, \\ \frac{N-1}{N-2}\max\{2,k\} & \text{for } p > 2 \end{cases} \qquad I > d - 1 + \frac{N-1}{N-2}\max\{2,k\}.$$

Thus one easily gets that T_ε has a unique fixed point in set

$$\mathcal{B} = \{\phi \in H_0^1(\Omega_\rho) \,:\, \|\phi\|_{H_0^1(\Omega_\rho)} \leq \varepsilon^d\},$$

as a direct application of the contraction mapping Theorem. This concludes the proof.

5. The linear theory: proof of Proposition 4.1

In this section, we will establish a solvability theory for the linear problem to prove Proposition 4.1. We first study the above problem in a strip close to the scaled manifold K_ρ. Let $\gamma \in (\frac{1}{2}, 1)$ be the number fixed before in (4.3) and define

$$\Omega_{\rho,\gamma} := \{x \in \Omega_\rho \,:\, \mathrm{dist}(x, K_\rho) < 2\varepsilon^{-\gamma}\}. \tag{5.1}$$

We are first interested in solving the following problem: given $f \in L^2(\Omega_{\rho,\gamma})$

$$-\Delta\phi - (p-\varepsilon)\tilde{U}_\varepsilon^{p-1-\varepsilon}\phi = f \quad \text{in } \Omega_{\rho,\gamma}, \quad \phi = 0 \quad \text{on } \partial\Omega_{\rho,\gamma}. \tag{5.2}$$

We have the validity of the following result.

Proposition 5.1. *There exist a constant $C > 0$ and a sequence $\varepsilon_l = \varepsilon \to 0$ such that, for any $f \in L^2(\Omega_{\rho,\gamma})$ there exists a solution $\phi \in H_0^1(\Omega_{\rho,\gamma})$ to Problem (5.2) such that*

$$\|\phi\|_{H_0^1(\Omega_{\rho,\gamma})} \leq C\rho^{-\max\{2,k\}}\|f\|_{L^2(\Omega_{\rho,\gamma})}. \tag{5.3}$$

Proof. Consider the functional

$$E(\phi) = \frac{1}{2}\int_{\Omega_{\rho,\gamma}} (|\nabla\phi|^2 - (p-\varepsilon)\tilde{U}_\varepsilon^{p-1-\varepsilon}\phi^2) \tag{5.4}$$

for functions $\phi \in H^1(\Omega_{\rho,\gamma})$. Let $(y,x) \in \mathbb{R}^{k+N}$ be the local coordinates along K_ρ. With an abuse of notation we will denote

$$\phi(\Upsilon(y,x)) = \phi(z,x), \quad \text{with} \ \ y = \rho z. \tag{5.5}$$

Since the original variable $(z,x) \in \mathbb{R}^{k+N}$ are only local coordinates along K_ρ we let the variable (z,x) vary in the set $\mathcal{C}_\varepsilon$ defined by

$$\mathcal{C}_\varepsilon = \{(z,x) \ / \ \rho z \in \ K, \quad |x| < \varepsilon^{-\gamma}\}. \tag{5.6}$$

We write $\mathcal{C}_\varepsilon = \frac{1}{\rho}K \times \hat{\mathcal{C}}_\varepsilon$ where

$$\hat{\mathcal{C}}_\varepsilon = \{x \ / \ |x| < \varepsilon^{-\gamma}\}. \tag{5.7}$$

Observe that $\hat{\mathcal{C}}_\varepsilon$ approaches, as $\varepsilon \to 0$, the whole space $\mathbb{R}^N$.

In these new local coordinates, the energy density associated to the functional E in (5.4) is given by

$$\frac{1}{2}\left[|\nabla\phi|^2 - (p-\varepsilon)\tilde{U}_\varepsilon^{p-1-\varepsilon}\phi^2\right]\sqrt{\det(g^\varepsilon)}, \tag{5.8}$$

where ∇_{g^ε} denotes the gradient in the new variables and where g^ε is the metric in the coordinates (z,x). Arguing as in [15], we have that, if (z,x) vary in $\mathcal{C}_\varepsilon$, then, the energy functional (5.4) in the new variables (5.5) is given by

$$\begin{aligned} E(\phi) \ \ = \ \ & \int_{K_\rho\times\hat{\mathcal{C}}_\varepsilon}\left(\frac{1}{2}(|\nabla_x\phi|^2 - (p-\varepsilon)\tilde{U}_\varepsilon^{p-1-\varepsilon}\phi^2)\right)\sqrt{\det(g^\varepsilon)}\,dz\,dx \\ & + \int_{K_\rho\times\hat{\mathcal{C}}_\varepsilon}\frac{1}{2}\,\Xi_{ij}(\rho z,x)\,\partial_i\phi\partial_j\phi\,\sqrt{\det(g^\varepsilon)}\,dz\,dx \\ & + \frac{1}{2}\int_{K_\rho\times\hat{\mathcal{C}}_\varepsilon}|\nabla_{K_\rho}\phi|^2\,\sqrt{\det(g^\varepsilon)}\,dz\,dx + \int_{K_\varepsilon\times\hat{\mathcal{C}}_\varepsilon}B(\phi,\phi)\,\sqrt{\det(g^\varepsilon)}\,dz\,dx, \end{aligned} \tag{5.9}$$

where

$$\Xi_{ij}(\rho z,x) = 2\rho H_{ij}x_N - \frac{\rho^2}{3}\,R_{islj}x_lx_s - \rho^2x_N^2(H^2)_{ij}, \tag{5.10}$$

and we denoted by $B(\phi,\phi)$ a quadratic term in ϕ that can be expressed in the following form

$$B(\phi,\phi) = O\left(\rho^3|x|^3\right)\partial_i\phi\partial_j\phi + \rho\,|\nabla_{K_\varepsilon}\phi|^2\,O(\rho^2|x|) + \partial_j\phi\partial_{\bar a}\phi\,(\mathcal{O}(\rho|x|)) \tag{5.11}$$

and we used the Einstein convention over repeated indices. Furthermore we used the notation $\partial_a = \partial_{y_a}$ and $\partial_{\bar a} = \partial_{z_a}$.

Given a function $\phi \in H^1(\Omega_{\rho,\gamma})$, we decompose it as

$$\phi = \left[\frac{\delta}{\mu_\varepsilon}\tilde{\mathcal{T}}_{\mu_\varepsilon,d_\varepsilon}(Z_{N+1}) + \sum_{j=1}^N\frac{d^j}{\mu_\varepsilon}\tilde{\mathcal{T}}_{\mu_\varepsilon,d_\varepsilon}(Z_j) + \frac{e}{\mu_\varepsilon}\tilde{\mathcal{T}}_{\mu_\varepsilon,d_\varepsilon}(Z_0)\right]\bar\chi_\varepsilon + \phi^\perp \tag{5.12}$$

where the expression $\tilde{\mathcal{T}}_{\mu_\varepsilon,d_\varepsilon}(v)$ is defined in (4.4), the functions Z_{N+1} and Z_j are already defined in (3.3) and where Z_0 is the eigenfunction, with $\int_{\mathbb{R}^N}Z^2 = 1$, corresponding to the unique positive eigenvalue λ_1 in $L^2(\mathbb{R}^N)$ of the problem

$$\Delta_{\mathbb{R}^N}\phi + pU^{p-1}\phi = \lambda_1\phi \quad \text{in} \quad \mathbb{R}^N. \tag{5.13}$$

It is worth mentioning that $Z_0(\xi)$ is even and it has exponential decay of order $O(e^{-\sqrt{\lambda_1}|\xi|})$ at infinity. The function $\bar\chi_\varepsilon$ is a smooth cut off function defined by

$$\bar\chi_\varepsilon(x) = \hat\chi_\varepsilon\left(\left|\left(\frac{\bar x - \varepsilon^2\rho^{-1}\bar d_\varepsilon}{\mu_\varepsilon}, \frac{x_N - \varepsilon\rho^{-1}d_{N,\varepsilon}}{\mu_\varepsilon}\right)\right|\right), \tag{5.14}$$

with $\hat\chi(r) = 1$ for $r \in (0, \frac{3}{2}\varepsilon^{-\gamma})$, and $\chi(r) = 0$ for $r > 2\varepsilon^{-\gamma}$. Finally, in (5.12) we have that $\delta = \delta(\rho z)$, $d^j = d^j(\rho z)$ and $e = e(\rho z)$ are function defined in K such that $\forall z \in K_\rho$

$$\int_{\hat{\mathcal{C}}_\varepsilon} \phi^\perp \tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(Z_{N+1})\bar\chi_\varepsilon dx = \int_{\hat{\mathcal{C}}_\varepsilon} \phi^\perp \tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(Z_j)\bar\chi_\varepsilon = \int_{\hat{\mathcal{C}}_\varepsilon} \phi^\perp \tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(Z_0)\bar\chi_\varepsilon = 0. \tag{5.15}$$

We will denote by $(H^1_\varepsilon)^\perp$ the subspace of the functions in H^1_ε that satisfy the orthogonality conditions (5.15).

A direct computation shows that

$$\delta(\rho z) = \frac{\int \phi \tilde{\mathcal{T}}_{\mu_\varepsilon,d_\varepsilon}(Z_{N+1})}{\mu_\varepsilon \int Z^2_{N+1}}(1 + O(\varepsilon)) + O(\varepsilon)(\sum_j d^j(\rho z) + e(\rho z)),$$

$$d^j(\rho z) = \frac{\int \phi \tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(Z_j)}{\mu_\varepsilon \int Z^2_j}(1 + O(\varepsilon)) + O(\varepsilon)(\delta(\rho z) + \sum_{i\neq j} d^i(\rho z) + e(\rho z)),$$

and

$$e(\rho z) = \frac{\int \phi \tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(Z_0)}{\mu_\varepsilon \int Z^2_0}(1 + O(\varepsilon)) + O(\varepsilon)(\delta(\rho z) + \sum_j d^j(\rho z)).$$

Observe that, since $\phi \in H^1_{g^\varepsilon}$, one easily get that the functions δ, d^j and e belong to the Hilbert space

$$\mathcal{H}^1(K) = \{\zeta \in \mathcal{L}^2(K) : \partial_a \zeta \in \mathcal{L}^2(K), \quad a = 1, \cdots, k\}. \tag{5.16}$$

Observe that in the region we are considering the function $\tilde U_\varepsilon$ is nothing but $\tilde U_\varepsilon = \tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(v_{I+1,\varepsilon})$, where $v_{I+1,\varepsilon}$ is the function whose existence and properties are proven in Lemma 3.2. For the argument in this part of our proof it is enough to take $I = 3$, and for simplicity of notation we will denote by $\hat w$ the function $v_{I+1,\varepsilon}$ with $I = 3$. Referring to (3.26) we have

$$\hat w(z, \xi) = U(\xi) - \bar U(\xi) + \sum_{i=1}^{4} w_{i,\varepsilon}(z, \xi) \tag{5.17}$$

where $U = w_N$ and $\bar U$ are defined in (1.4) and (3.21), and

$$\|D^2_\xi w_{i+1,\varepsilon}\|_{\varepsilon,N-2,\sigma} + \|D_\xi w_{i+1,\varepsilon}\|_{\varepsilon,N-3,\sigma} + \|w_{i+1,\varepsilon}\|_{\varepsilon,N-4,\sigma} \le C\varepsilon^{i+1} \tag{5.18}$$

and, for any integer ℓ

$$\|\nabla^{(\ell)}_y w_{i+1,\varepsilon}(y, \cdot)\|_{\varepsilon,N-2,\sigma} \le \beta C_l \varepsilon^{i+1} \qquad y = \rho z \in K$$

for any $i = 0, 1, 2, 3$.

Thanks to the above decomposition (5.12), we have the validity of the following expansion for $E(\phi)$.

$$\begin{aligned} E(\frac{\delta}{\mu_\varepsilon}\tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(Z_{N+1})\bar\chi_\varepsilon) = \rho^{-k}\varepsilon\frac{1}{2}\int_K \Bigg[& A_{1,\varepsilon}\varepsilon^{1+\frac{2}{N-2}}|\nabla_K(\delta(1 + o(\varepsilon)\beta^\varepsilon_1(y)))|^2 - (N-2)A_1\frac{\mu_0^{N-4}}{(d^0_N)^{N-2}}\delta^2 \\ & +(N-2)A_1\frac{\mu_0^{N-3}}{(d^0_N)^{N-1}}\delta d_N + \varepsilon^{\frac{1}{N-2}}\frac{\delta}{\mu_0}\left(\frac{\mu_0}{d^0_N}\right)^{N-1} g_{N+1}\left(\frac{\mu_0}{d^0_N}\right)\Bigg] dz \end{aligned} \tag{5.19}$$

$$\begin{aligned} E(\frac{d_N}{\mu_\varepsilon}\tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(Z_N)\bar\chi_\varepsilon) = \rho^{-k}\rho^2\frac{1}{2}\int_K \Bigg[& A_{2,\varepsilon}\varepsilon|\nabla_K(d_N(1 + o(\varepsilon)\beta^\varepsilon_2(y)))|^2 - (N-2)A_1\frac{\mu_0^{N-3}}{(d^0_N)^{N-1}}\delta d_N \\ & +(N-1)A_3\frac{\mu_0^{N-2}}{(d^0_N)^N}d^2_N + \varepsilon^{\frac{1}{N-2}}\frac{d_N}{\mu_0}\left(\frac{\mu_0}{d^0_N}\right)^N g_N\left(\frac{\mu_0}{d^0_N}\right)\Bigg] dz \end{aligned} \tag{5.20}$$

$$
\begin{aligned}
&E(\frac{d_j}{\mu_\varepsilon}\tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(Z_j)\bar{\chi}_\varepsilon) \qquad (5.21)\\
= \ &\rho^{-k}\rho^2\frac{1}{2}\int_K \left[A_{3,\varepsilon}|\nabla_K(d_j(1+o(\varepsilon)\beta_3^\varepsilon(y)))|^2 - \frac{\mathcal{R}_{mj}}{4}d_j d_m + \varepsilon^{\frac{1}{N-2}}\frac{d_j}{\mu_0}\left(\frac{\mu_0}{d_N^0}\right)^{N-1} g_j\left(\frac{\mu_0}{d_N^0}\right)\right]dz
\end{aligned}
$$

$$
\begin{aligned}
&E(\frac{e}{\mu_\varepsilon}\tilde{\mathcal{T}}_{\mu_\varepsilon, d_\varepsilon}(Z_0)) \qquad (5.22)\\
= \ &\rho^{-k}\frac{1}{2}\int_K \left[D_1\,|\partial_a e + e^{-\sqrt{\lambda_1}\varepsilon^{-\gamma}}\beta_4^\varepsilon(y)e|^2 - \lambda_1 D_1 e^2 - D_2 d_N^0 e\right]\left(1+\varepsilon O(e^{-\sqrt{\lambda_1}|\xi|})\right).
\end{aligned}
$$

Therefore, μ and $d_1, \cdots, d_{N-1}, d_N$ and e satisfy

$$
\begin{cases}
L_{N+1}(\delta, d_N) := -c_1\varepsilon^{1+\frac{2}{N-2}}\mu_0\Delta_K\delta + A\delta + Bd_N = \alpha_{N+1} + \varepsilon M_{N+1};\\
L_N(\delta, d_N) := -c_2\varepsilon\mu_0\Delta_K d_N + B\delta + Cd_N = \alpha_N + \varepsilon M_N;\\
L_j(\bar{d}) := -\Delta_K d_j + \left(\tilde{g}^{ab}R_{mabj} - \Gamma_a^c(E_m)\Gamma_c^a(E_j)\right)d_m = \alpha_j + \varepsilon M_j,\ j = 1, \dots, N-1;\\
L_0(e) := \Delta_K e + D_1\lambda_1 e + D_2 d_N = \alpha_0 + \varepsilon Q_0 + \varepsilon^2 M_0,
\end{cases}
\tag{5.23}
$$

where

$$
A = -(N-2)A_1\frac{\mu_0^{N-3}}{(d_N^0)^{N-2}}, \quad B = (N-2)A_1\frac{\mu_0^{N-2}}{(d_N^0)^{N-1}}, \quad C = -(N-1)A_3\frac{\mu_0^{N-1}}{(d_N^0)^{N}},
$$

with $AC - B^2 > 0$, and

$$
D_1 = \int_{\mathbb{R}^N} Z_0^2(\xi)\,d\xi, \quad \text{and} \qquad D_2 = 2H_{jj}d_{N,\varepsilon}^0\int_{\mathbb{R}^N}\partial_{jj}^2 U Z_0 d\xi.
$$

The functions α_j are explicit function of z in K,smooth and uniformly bounded in ε. The operators $M_i = M_i(\mu, d, e)$ can be decomposed in the following form

$$
M_i(\mu, d, e) = A_i(\mu, d, e) + K_i(\mu, d, e)
$$

where K_i is uniformly bounded in $L^\infty(K)$ for (μ, d, e) and is also compact. The operator A_i depends on (μ, d, e) and their first and second derivatives and it is Lipschitz in this region, that is

$$
\|A_i(\mu_1, d_1, e_1) - A_i(\mu_2, d_2, e_2)\|_\infty \le Co(1)\|(\mu_1-\mu_2, d_1-d_2, e_1-e_2)\|.
$$

We remark that the dependence on $\ddot{\mu}$, $\ddot{d}$ and $\ddot{e}$ is linear. Finally, the operator Q_0 is quadratic in d and it is uniformly bounded in $L^\infty(K)$ for (δ, d, e) satisfying (3.23)-(3.25)

Our goal is now to solve (5.23) in δ, d and e. To do so, we first analyze the invertibility of the linear operators L_i.

We start with a linear theory for the problem

$$
L_{N+1}(\delta, d_N) = h_1, \quad L_N(\delta, d_N) = h_2, \tag{5.24}
$$

with h_1 and h_2 bounded. Arguing as in the proof of Lemma 8.1 in [17] (with obvious modifications), we can prove that, assuming $A < 0$, $C < 0$ and $AC - B^2 > 0$ and that $\|h_1\|_\infty + \|h_2\|_\infty$ is bounded. Then there exist (μ, d) solution to the above system and a constant C such that

$$
\|\mu\|_\infty + \|d_N\|_\infty + \varepsilon^{\frac{1}{2}+\frac{1}{N-2}}\|\nabla_K\mu\|_\infty + \varepsilon^{\frac{1}{2}}\|\nabla_K d_N\|_\infty \le C\left[\,\|h_1\|_\infty + \|h_2\|_\infty\,\right]. \tag{5.25}
$$

As we mentioned above, to obtain this we follows the lines of the proof of Lemma 8.1 in [17]. For existence we use the fact that the system (5.24) has a variational structure with associated energy functional

$$
J(\delta, d_N) = \frac{1}{2}c_1\varepsilon^{1+\frac{2}{N-2}}\mu_0\int_K|\nabla_K\delta|^2 + c_2\varepsilon\mu_0\int_K|\nabla_K d_N|^2 + \frac{1}{2}(A\int_K\delta^2 + 2B\int_K\delta d_N + C\int_K d_N^2)
$$

and clearly by our assumption on the constants A, B, C this energy functional is positive, bounded from below away from zero and convex. Then, existence of solution follows. The a-priori estimate (5.25) follows by a contradiction argument (as in Lemma 8.1 in [17]). Indeed, if (5.25) is false, we have existence of a sequence (h_{1n}, h_{2n}) with $\|h_{1n}\|_\infty + \|h_{2n}\|_\infty \to 0$, and a sequence of solutions $(\delta_n, (d_N)_n)$ with

$$\|\delta_n\|_\infty + \|(d_N)_n\|_\infty + \varepsilon^{\frac{1}{2}+\frac{1}{N-2}}\|\nabla_K \delta_n\|_\infty + \varepsilon^{\frac{1}{2}}\|\nabla_K (d_N)_n\|_\infty = 1.$$

Since $A < 0$ and $C < 0$ and $C - \frac{B^2}{A} > 0$ and applying the maximum principle, Ascoli-Arzelá theorem we end up with a contradiction. Now for every $j = 1, \dots, N-1$ the operator L_j is invertible by the non degeneracy of the submanifold K. We can then prove that the equation $L_j\bar{d} = f$ is solvable on $\bar{d}$ and the following estimate holds true

$$\|\bar{d}\|_\infty + \|\partial_a \bar{d}\|_\infty + \|\partial^2_{ab}\bar{d}\|_\infty \le C\,\|f\|_\infty\,. \tag{5.26}$$

We are then left with the study of the invertibility of the operator L_0. we prove it as the following result. □

Lemma 5.1. *There is a sequence $\varepsilon = \varepsilon_j \searrow 0$ such that for any $\varphi \in C^{0,\alpha}(K)$, there exists a unique $e \in C^{2,\alpha}(K)$ such that*

$$L_0(e) = \varphi \tag{5.27}$$

with the property

$$\|e\|_* := \|e\|_{L^\infty(K)} + \rho\|\nabla e\|_{L^\infty(K)} + \rho^2\|\nabla^2 e\|_{L^\infty(K)} \le C\rho^{-k}\|\varphi\|_{L^\infty(K)}, \tag{5.28}$$

where C is a positive constant independent of ε_j.

Proof. The proof is classical, the arguments are similar in spirit to the ones used in [15], [24] and some references therein. We also refer the reader to the papers [25, 30] for a different setting. So we will omit the proof here. □

Proof of Proposition 4.1: Using Proposition 5.1, we can get the existence of solutions to the linear problem in the whole domain Ω_ρ, we refer the reader to [15] for the detail proof.

6. Appendix A

Proofs of (3.31)-(3.33): We recall that $h_{1,\varepsilon} = h_{11} + \varepsilon h_{12} + \rho h_{13}$ where we have set

$$\begin{aligned} h_{11} &= pU^{p-1}\bar{U} + \varepsilon U^p \log U, \\ h_{12} &= -\frac{N-2}{2}U^p \log(\mu_{0,\varepsilon}) - 2d^0_{N,\varepsilon}H_{ij}\partial^2_{ij}U - \lambda_1 e_{0,\varepsilon}Z_0, \\ h_{13} &= \mu_{0,\varepsilon}\left[-2\xi_N H_{ij}\partial^2_{ij}U + H_{\alpha\alpha}\partial_N U\right]. \end{aligned}$$

By the result of [17], we have

$$\int_{\hat{\mathcal{D}}} h_{11}Z_{N+1}d\xi = \varepsilon\left[-A_1\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-2} + A_2 + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1} g_{N+1}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)\right] \tag{6.1}$$

$$\int_{\hat{\mathcal{D}}} h_{11}Z_N d\xi = \varepsilon^{1+\frac{1}{N-2}}\left[A_3\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1} + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N} g_N\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)\right] \tag{6.2}$$

$$\int_{\hat{\mathcal{D}}} h_{11}Z_l d\xi = \varepsilon^{2+\frac{3}{N-2}} g_l\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right) \qquad \text{for } \ l = 1, \dots, N-1, \tag{6.3}$$

$$\int_{\hat{\mathcal{D}}} h_{11}Z_0 d\xi = \varepsilon\left[A_4\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-2} + A_5 + \varepsilon^{\frac{1}{N-2}}\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)^{N-1} g_0\left(\frac{\mu_{0,\varepsilon}}{d^0_{N,\varepsilon}}\right)\right] \tag{6.4}$$

where the functions g_i are smooth function with $g_i(0) \neq 0$ and A_i are positive constants. In particular, $A_3 = \frac{p\alpha_N^{\frac{N+2}{2}}(N-2)^2}{2^{N-1}}(\int \frac{\xi_N^2}{(1+|\xi|^2)^{\frac{N+4}{2}}})d\xi$.

It remain to compute h_{12} and h_{13} product with Z_i for $i = 0, 1, \dots, N+1$. First, by symmetry, we have

$$\int_{\hat{\mathcal{D}}} (\varepsilon h_{12} + \rho h_{13}) Z_l d\xi = \varepsilon^{2+\frac{3}{N-2}} \Theta \qquad \text{for } \ l = 1, \dots, N-1, \tag{6.5}$$

where Θ denotes a sum of functions of the form

$$\begin{aligned} & f_1(\rho z)\left[f_2(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}, \partial_a \mu_{0,\varepsilon}, \partial_a d^0_{N,\varepsilon}, \partial_e e_{0,\varepsilon}) + \right. \\ & \qquad \left. + o(1) f_3(\mu_{0,\varepsilon}, d^0_{N,\varepsilon}, e_{0,\varepsilon}, \partial_a \mu_{0,\varepsilon}, \partial_a d^0_{N,\varepsilon}, \partial_a e_{0,\varepsilon}, \partial^2_{aa} \mu_{0,\varepsilon}, \partial^2_{aa} d^0_{N,\varepsilon}, \partial^2_{aa} e_{0,\varepsilon})\right] \end{aligned} \tag{6.6}$$

where f_1 is a smooth function uniformly bounded in ε, f_2 and f_3 are smooth functions of their arguments, uniformly bounded in ε as $\mu_{0,\varepsilon}$, $d^0_{N,\varepsilon}$ and $e_{0,\varepsilon}$ are uniformly bounded, and $o(1) \to 0$ as $\varepsilon \to 0$.

First, taking product with Z_{N+1}, we have

$$\begin{aligned} \int_{\hat{\mathcal{D}}} (\varepsilon h_{12} + \rho h_{13}) Z_{N+1} d\xi &= \varepsilon \int_{\hat{\mathcal{D}}} h_{12} Z_{N+1} d\xi + \varepsilon^2 \Theta \\ = \quad \varepsilon \int_{\hat{\mathcal{D}}} & \left\{ -\frac{N-2}{2} U^p \log(\mu_{0,\varepsilon}) - 2 d^0_{N,\varepsilon} H_{ij} \partial^2_{ij} U \right\} Z_{N+1} d\xi + \varepsilon^2 \Theta \end{aligned}$$

where Θ is a sum of functions of the form (6.6).

We set $U_\lambda(\xi) = \alpha_N \left(\frac{\lambda}{\lambda^2 + |\xi|^2}\right)^{\frac{N-2}{2}}$. Since $(\partial_\lambda U_\lambda)_{|\lambda=1} = -Z_{N+1}$, we have

$$\int_{\hat{\mathbb{R}^N}} U^p Z_{N+1} = \int_{\mathbb{R}^N} U^{\frac{N+2}{N-2}} Z_{N+1} = -\frac{N-2}{2N} \partial_\lambda \left(\int_{\mathbb{R}^N} U_\lambda^{\frac{2N}{N-2}} \right)_{|\lambda=1} = 0.$$

Here we used the fact that and $\int_{\mathbb{R}^N} U_\lambda^{\frac{2N}{N-2}}$ does not depend on λ (by simple change of variables argument). Moreover, a direct computation gives

$$H_{ij} \int_{\mathbb{R}^N} \partial^2_{ij} U Z_{N+1} d\xi = 0$$

Collecting these facts, we get $\int_{\hat{\mathcal{D}}} (\varepsilon h_{12} + \rho h_{13}) Z_{N+1} d\xi = \varepsilon^2 \Theta$, where Θ is a sum of functions of the form (6.6).

Next, taking product with Z_N, we have

$$\begin{aligned} \int_{\hat{\mathcal{D}}} (\varepsilon h_{12} + \rho h_{13}) Z_N d\xi &= \rho \int_{\hat{\mathcal{D}}} h_{13} Z_N d\xi + \varepsilon^2 \Theta \\ &= \rho \mu_{0,\varepsilon} \left[-\int_{\mathbb{R}^N} 2\xi_N H_{ij} \partial^2_{ij} U \partial_N U d\xi + H_{\alpha\alpha} \int_{\mathbb{R}^N} |\partial_N U|^2 d\xi \right] + \varepsilon^2 \Theta \\ &= \rho \mu_{0,\varepsilon} \left[-(H_{jj} - H_{\alpha\alpha}) \int_{\mathbb{R}^N} |\partial_N U|^2 d\xi \right] + \varepsilon^2 \Theta \\ &= \rho \mu_{0,\varepsilon} H_{aa} \int_{\mathbb{R}^N} |\partial_N U|^2 d\xi + \varepsilon^2 \Theta, \end{aligned}$$

where Θ is a sum of functions of the form (6.6). Here we used the following fact

$$\begin{aligned}
&\int_{\mathbb{R}^N} \xi_N H_{ij}\partial^2_{ij}U\partial_N U d\xi = H_{jj}\int_{\mathbb{R}^N} \xi_N \partial^2_{jj}U\partial_N U d\xi \\
= \;& \frac{1}{N-1}H_{jj}\int_{\mathbb{R}^N} \xi_N\partial_N U\sum_{i=1}^{N-1}\partial^2_{ii}U d\xi \\
= \;& \frac{1}{N-1}H_{jj}\int_{\mathbb{R}^N} \xi_N\partial_N U(\Delta U-\partial^2_{NN}U)d\xi \\
= \;& -\frac{1}{N-1}H_{jj}\int_{\mathbb{R}^N} \xi_N\partial_N U(U^p+\partial^2_{NN}U)d\xi \\
= \;& -\frac{1}{N-1}H_{jj}\left[\frac{1}{p+1}\int_{\mathbb{R}^N}\xi_N\partial_N(U^{p+1})d\xi+\frac{1}{2}\int_{\mathbb{R}^N}\xi_N\partial_N(|\partial_N U|^2)d\xi\right] \\
= \;& \frac{1}{N-1}H_{jj}\left[\frac{1}{p+1}\int_{\mathbb{R}^N}U^{p+1}d\xi+\frac{1}{2}\int_{\mathbb{R}^N}|\partial_N U|^2 d\xi\right] \\
= \;& \frac{1}{N-1}H_{jj}\left[\frac{1}{p+1}N\int_{\mathbb{R}^N}|\partial_N U|^2 d\xi+\frac{1}{2}\int_{\mathbb{R}^N}|\partial_N U|^2 d\xi\right] \\
= \;& \frac{1}{2}H_{jj}\int_{\mathbb{R}^N}|\partial_N U|^2 d\xi,
\end{aligned}$$

since $\int_{\mathbb{R}^N} U^{p+1}d\xi = \int_{\mathbb{R}^N}(-\Delta U)U d\xi = \int_{\mathbb{R}^N}|\nabla U|^2 d\xi = N\int_{\mathbb{R}^N}|\partial_N U|^2 d\xi$.

Finally, using the orthogonality in L^2 of Z_0 with respect to Z_i, for $i = 1,\dots,N+1$, a direct computations show

$$\int_{\hat{\mathcal{D}}}(\varepsilon h_{12}+\rho h_{13})Z_0 d\xi = -A_7\log(\mu_{0,\varepsilon})-\lambda_1 e_{0,\varepsilon}-2H_{jj}d^0_{N,\varepsilon}\int_{\mathbb{R}^N}\partial^2_{jj}U Z_0 d\xi+\varepsilon^2\Theta$$

where Θ is a sum of functions of the form (6.6), and $A_7 = \frac{N-2}{2}\int_{\mathbb{R}^N}U^p Z_0 d\xi$.

Collecting all formulas from (6.1), we get the results.

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